RESIDUES AND POLES:

INTRODUCTION:

In mathematics, more specifically complex analysis, the **residue** is a complex number proportional to the contour integral of a meromorphic function along a path enclosing one of its singularities. Residues can be computed quite easily and, once known, allow the determination of general contour integrals via the residue theorem.

The **residue theorem**, sometimes called **Cauchy's Residue Theorem**, in complex analysis is a powerful tool to evaluate line integrals of analytic functions over closed curves and can often be used to compute real integrals as well. It generalizes the Cauchy integral theorem and Cauchy's integral formula.

SINGULAR POINT:

A point z_0 is called a singular point of a function f(z) is failed to be analytic at z_0 but it is analytic in some point in every neighborhood.

ISOLATED SINGULAR POINT:

A singular point z_0 is said to be isolated singular point if there is a deleted neighborhood

$0 \leq | z-z_0 |<£$ of z_0 throughout of which f is analytic.

RESIDUE:

The residue of a meromorphic function f at an isolated singularity a, often denoted $\operatorname{Res}(f, a)$ is the unique value R such that $f(z) - R / (z - a)$ has an analytic antiderivative in a punctured disk $0 < |z - a| < \delta$. Alternatively, residues can be calculated by finding Laurent series expansions, and one can define the residue as the coefficient a_{-1} of a Laurent series.

DEFINITION:

The complex number under b_1 is a co efficient of $1/(z - z_0)$ is called Residue of at hte isolated singular point z_0 . we used the notation Residue of f(z) = b_1=B.

$$b_1 = \frac{1}{2\pi i} \int^{z=z_0} f(z)\, dz$$

$$= \int \quad f(z)\, dz = 2\pi i b_1$$

EXAMPLE:

Consider the contour integral

Where C is some simple closed curve about 0.

Let us evaluate this integral without using standard integral theorems that may be available to us. Now, the Taylor series for e^z is well-known, and we substitute this series into the integrand. The integral then becomes

Let us bring the $1/z^5$ factor into the series, so we obtain

The integral now collapses to a much simpler form. Recall that

So now the integral around C of every other term not in the form cz^{-1} becomes zero, and the integral is reduced to

EXAMPLE:

Consider the integral.

$$\int_C \frac{dz}{z(z-2)^4}$$

Where C is the positively oriented circle $|z-2| = 1$. Since the integrand is analytic every where in the finite plane except at the points z=0 and z=2, it has a Laurent series representation that is valid in the punctured disk $0 < |z-2| < 2$, thus by residue method.

$$\frac{1}{(z-2)^4}\,\frac{1}{(z+2-2)}$$

$$= \frac{1}{2(z-2)^4}\,\frac{1}{(1+\frac{z-2}{2})}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}(z-2)^{n-4} \quad , 0 < |z-2| < 2$$

In this Laurent series, which could be written in the form $f(z) = \sum_{n=-\infty}^{\infty} a_n\,(z-z_0)^n$,the co efficient of $1/(z-2)$ is the desired residue, namely -1/16. Consequently,

$$\int_C \frac{dz}{z(z-2)^4} = 2\pi i\left(-\frac{1}{16}\right) = -\frac{\pi i}{8}$$

CAUCHY’S RESIDUE THEROEM:

STATEMENT

Let us consider C be a simple closed contour, described in the positive sense. If a function f is analytic inside and on C except for a finite number of singular points z_k (k=1,2,3,…..,n) inside C , then

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} \underset{z=z_k}{Res} f(z) \qquad \ldots\ldots\ldots\ldots\ldots\ (1)$$

PROOF:

Let the points z_k (k=1,2,3,…..,n) be centers of positively oriented circles C_k which are interior to C and are so small that no two of them have points in common. The circle C_k , together with the simple closed contour C, from the boundary of a closed region throughout which f is analytic and whose interior is a multiply connected domain. Hence according to the extension of the Cauchy-Goursat theorem to such regions

$$\int_C f(z)dz = \sum_{k=1}^{n} \int_{C_k} f(z)dz = 0$$

This reduces to equation (1)

$$\int_{C_k} f(z)dz = 2\pi i \sum_{k=1}^{n} \underset{z=z_k}{Res} f(z) \qquad (k = 1,2,3, \ldots \ldots, n).$$

And the proof is complete.

EXAMPLE

Suppose $t > 0$ and define the contour C that goes along the real line from $-a$ to a and then counterclockwise along a semicircle centered at 0 from a to $-a$. Take a to be greater than 1, so that the imaginary unit i is enclosed within the curve. The contour integral is

Since e^{itz} is an entire function (having no singularities at any point in the complex plane), this function has singularities only where the denominator $z^2 + 1$ is zero. Since $z^2 + 1 = (z + i)(z - i)$, that happens only where $z = i$ or $z = -i$. Only one of those points is in the region bounded by this contour. Because $f(z)$ is

the residue of $f(z)$ at $z = i$ is

According to the residue theorem, then, we have

EXAMPLE:

Let us use the theorem to use the integral.

$$\int_C \frac{5z-2}{z(z-1)}dz$$

When C is the circle $|z| = 2$, described counter lockewise. The integrand has the two isolated singularities z =0 and z=1. Both of which are interior to C. we can find the residues B_1 at z= 0 and B_2 at z =1 with the aid of the maclarian series

$\frac{1}{1-z} = 1 + z + z^2 + \ldots.$ $(|z| < 1).$

When $0 < |z| < 1$,

$\frac{5z-2}{z(z-1)} = \frac{5z-2}{z}\frac{-1}{1-z} = \left(5 - \frac{2}{z}\right)(-1 - z - z^2 - \cdots)$

And by identifying the coefficients of 1/z in the product on the right here ,we find that

B_1=2. Also since

$$\frac{5z-2}{z(z-1)} = \frac{5(z-1)+3}{z-1}\frac{1}{1+(z-1)}$$

$$= \left(5 + \frac{3}{z-1}\right)(1 - (z-1) + (z-1)^2 - \cdots)$$

When $0 < |z-1| < 1$, it is clear that B_2 =3 thus

$$\int_C \frac{5z-2}{z(z-1)}dz = 2\pi i(B_1 + B_2) = 10\pi i$$