# SOLUTIONS: QUIZ 1

CONTROL OF MOBILE ROBOTS

February 7, 2013

## 1

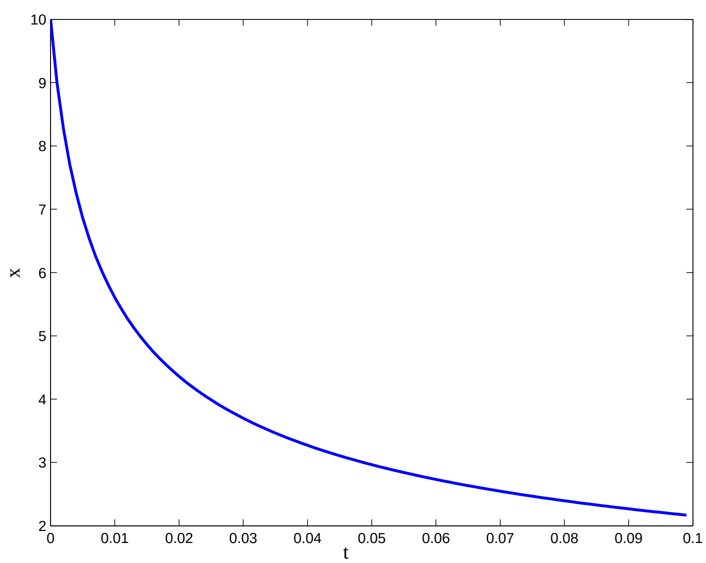
One way of getting a general feeling for what a differential equation is up to is to look at the sign and magnitude of the derivative at different points for different values of $x$. Use this idea for the dynamics

$$\dot{x}(t) = -x(t)^3.$$

Which one of the plots below (where $t$ is on the "x-axis" and $x(t)$ is on the "y-axis") was generated by this system?

## SOLUTION

We see that in all options, the state $x$ start at 10, i.e., $x(0) = 10$. But, $\dot{x} = -x^3$ means that the derivative has to be negative $-10^3$ at the beginning, i.e., $x$ has to be decreasing. And, as $x$ decreases, $x^3$ decreases as well, i.e., the derivative is going to remain negative (for positive $x$) but get smaller and smaller in magnitude, the smaller $x$ gets. As such, the only plot that fits this bill is:



## 2

One way of modeling epidemics is to describe how the fraction of infected individuals evolves over time. Let $I$ be that fraction, with the model being

$$\dot{I} = \beta I(1 - I) - \rho I.$$

Here, the constants $\beta$ and $\rho$ are the infection and recovery rates, respectively.

What are the possible equilibrium points to this system (values for $I$ when the fraction of infected individuals is not changing)?

# SOLUTION

An equlibrium point is a point where the system does not change. In other words, we need $\dot{I} = 0$. This means that

$$0 = \beta I(1 - I) - \rho I = I(\beta(1 - I) - \rho),$$

from which we see that $I = 0$ is one solution. The other solution is given by

$$\beta(1 - I) - \rho = 0 \;\Rightarrow\; 1 - I = \frac{\rho}{\beta} \;\Rightarrow\; I = 1 - \frac{\rho}{\beta} = (\beta - \rho)/\beta.$$

As such, the answer is $I = 0$ and $I = (\beta - \rho)/\beta$.

## 3

If someone gives you a possible solution to a differential equation, the way to check if this is indeed a solution is by taking the required number of derivatives and seeing if the proposed solution does in fact satisfy the differential equation.

Let

$$\ddot{x}(t) = -\omega^2 x(t).$$

Which of the following options is *not* a possible solution to this equation?

# SOLUTION

We have to check the different options:

$x = \cos(\omega t)$:

$$\dot{x} = -\omega \sin(\omega t) \;\Rightarrow\; \ddot{x} = -\omega^2 \cos(\omega t) = -\omega^2 x,$$

i.e., this is a solution.

$x = 0$:

$$\dot{x} = -0 \;\Rightarrow\; \ddot{x} = -0 = -\omega^2 0 = 0,$$

i.e., this is a solution.

$x = e^{-\omega t}$:

$$\dot{x} = -\omega e^{-\omega t} \;\Rightarrow\; \ddot{x} = \omega^2 e^{-\omega t} \neq -\omega^2 x,$$

i.e., this is not a solution.

$x = \sin(\omega t)$:

$$\dot{x} = \omega \cos(\omega t) \;\Rightarrow\; \ddot{x} = -\omega^2 \sin(\omega t) = -\omega^2 x,$$

i.e., this is a solution.

$x = \omega \sin(\omega t) - \cos(\omega t)$:

$$\dot{x} = \omega^2 \cos(\omega t) + \omega \sin(\omega t) \;\Rightarrow\; \ddot{x} = -\omega^3 \sin(\omega t) + \omega^2 \cos(\omega t) = -\omega^2 x,$$

i.e., this is a solution.

So, the only non-solution is $e^{-\omega t}$.

## 4

We saw that the model of a cruise-controller could be given by

$$\dot{x} = \frac{c}{m} u - \gamma x,$$

where $u$ is the input, $x$ is the speed of the car, and $c, m, \gamma$ are constant parameters.

If there was no wind resistance in the cruise-control model ($\gamma = 0$), what would the steady-state values be for the velocity $x$ when using a pure $D$-regulator, i.e., when $u = k\dot{e} = k(\dot{r} - \dot{x}) = -k\dot{x}$ (since $r$ is constant)?

# SOLUTION

Plugging in the values gives that

$$\dot{x} = -\frac{ck}{m}\dot{x},$$

which no longer is a differential equation but an algebraic equation with solution $\dot{x} = 0$. this would mean that somehow, magically, $\dot{x} = 0$ right away which is clearly physically impossible. But, allowing for this to even be a possibility means that $x$ never changes i.e., $x(t) = x(0)$ for all values of $t$, i.e., it is impossible to say what $x(\infty)$ is without knowing the initial conditions. (Really, the whole equation is somewhat suspect to start with since we are setting the velocity to be a function of the velocity which makes sense in discrete time where the"new" velocity can be a function of the "old" velocity, but certainly not in continuous time where the velocity at time $t$ is somehow supposed to be a function of the velocity at time $t$?...) So, the answer is: Impossible to say!

## 5

Let a discrete-time system be given by

$$x_{k+1} = \max\{0, 5 - x_k\}.$$

If this system starts at $x_0 = 10$, what happens to the state of the system?

# SOLUTION

At the first step, we get

$$x_1 = \max\{0, 5 - 10\} = \max\{0, -5\} = 0.$$

At the next step we get

$$x_2 = \max\{0, 5 - 0\} = \max\{0, 5\} = 5.$$

Iterating in this way yields

$$\begin{aligned} x_3 &= \max\{0, 5-5\} = \max\{0, 0\} = 0 \\ x_4 &= \max\{0, 5-0\} = \max\{0, 5\} = 5 \\ &\vdots \end{aligned}$$

As such, this discrete-time system keeps switching between 0 and 5.