The Pythagorean Theorem Revisited

By Peter Ash

I Three-Dimensional Pythagorean Theorem

Theorem: Let ABCD be a right tetrahedron with $\angle ADB = \angle ADC = \angle BDC = 90°$. Let $\alpha, \beta, \gamma, \delta$ indicate the face opposite vertex A, B, C, and D respectively, and use the absolute value to represent the area of a face. Then $|\alpha|^2 + |\beta|^2 + |\gamma|^2 = |\delta|^2$.

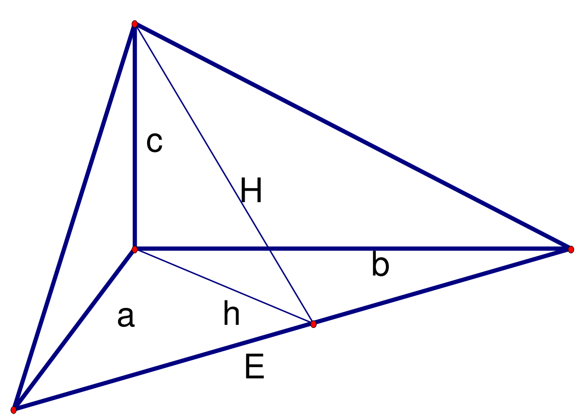
Proof 1 (Using Heron's Formula)

Let the lengths of the 3 edges meeting at D be a, b, and c. Then the sides of the triangle δ are $\sqrt{a^2+b^2}$, $\sqrt{b^2+c^2}$, and $\sqrt{a^2+c^2}$. Heron's formula yields

$$|\delta|^2 = s(s-\sqrt{a^2+b^2})(s-\sqrt{b^2+c^2})(s-\sqrt{a^2+c^2})$$

where $s = \frac{\sqrt{a^2+b^2}+\sqrt{b^2+c^2}+\sqrt{a^2+c^2}}{2}$. A "little" algebraic manipulation shows that $|\delta|^2 = \left(\frac{ab}{2}\right)^2 + \left(\frac{bc}{2}\right)^2 + \left(\frac{ac}{2}\right)^2 = |\gamma|^2 + |\alpha|^2 + |\beta|^2$.

Proof 2 (Straightforward geometry, from Kevin Brown)



The area of δ is $\frac{1}{2}HE$, where $E = \sqrt{a^2+b^2}$ and H is the altitude of δ with respect to E. If h is the altitude of γ with respect to E, $|\gamma| = \frac{1}{2}ab = \frac{1}{2}hE$, so that

$$h = \frac{ab}{E}$$

and $H^2 = c^2 + h^2 = c^2 + \frac{a^2b^2}{a^2+b^2}$. Thus

$$\begin{aligned} |\delta|^2 &= \frac{1}{4}H^2E^2 \\ &= \frac{1}{4}\left(c^2 + \frac{a^2b^2}{a^2+b^2}\right)\left(a^2+b^2\right) \\ &= \frac{1}{4}(c^2a^2 + c^2b^2 + a^2b^2) \\ &= |\beta|^2 + |\alpha|^2 + |\gamma|^2 \end{aligned}$$

Proof 3 Linear algebra

Let D be the origin and the coordinates of the other 3 vertices be (a, 0, 0), (0,b,0), and (0,0,c). Then $|\delta| = \frac{1}{2}\|\mathbf{u}\times\mathbf{v}\|$ where $\mathbf{u} = (0,b,0) - (a,0,0) = (-a,b,0)$ and $\mathbf{v} = (0,0,c) - (a,0,0) = (-a,0,c)$

so that $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} = bc\mathbf{i} + ac\mathbf{j} + ab\mathbf{k}$ and

$$|\delta|^2 = \frac{1}{4}\left((bc)^2 + (ac)^2 + (ab)^2\right) = |\alpha|^2 + |\beta|^2 + |\gamma|^2$$

Generalization and Proof 4 (from Eli Maor's *The Pythagorean Theorem*)

Let Π be a plane in 3-space, and δ be a region in Π with finite area. Let α, β, and γ be projections of δ onto the yz-, xz-, and xy-planes, respectively. To avoid overusing the absolute value signs in the proof, we use $\|\bullet\|$ to denote the area of a set. then

$$\|\delta\|^2 = \|\alpha\|^2 + \|\beta\|^2 + \|\gamma\|^2$$

Proof: Let **n** = (a, b, c) be a unit normal to Π. Basic facts about projections tell us that $\|\alpha\| = |a|\|\delta\|, \|\beta\| = |b|\|\delta\|$, and $\|\gamma\| = |c|\|\delta\|$. Therefore

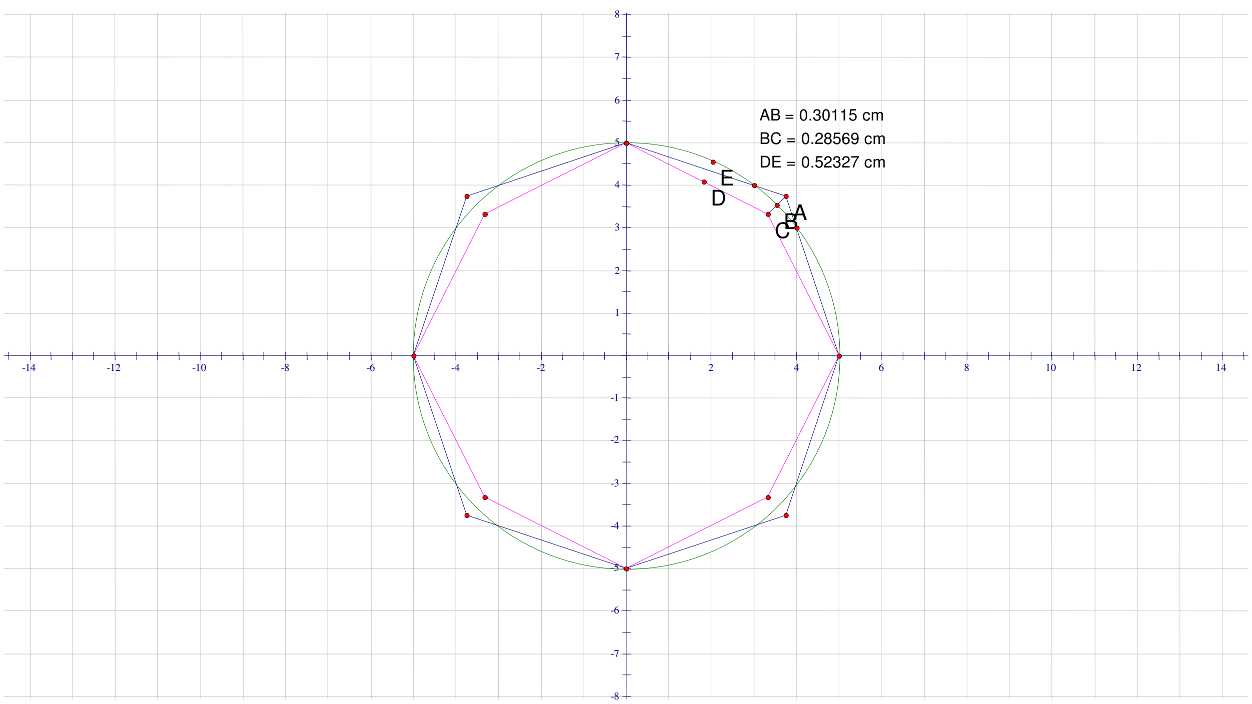
$$\|\alpha\|^2 + \|\beta\|^2 + \|\gamma\|^2 = a^2\|\delta\|^2 + b^2\|\delta\|^2 + c^2\|\delta\|^2$$
$$= \left(a^2 + b^2 + c^2\right)\|\delta\|^2$$
$$= \|\delta\|^2$$

This proof generalizes easily to an n-dimensional result.

II Approximation to Pythagorean Theorem

In computer graphics it's sometimes necessary to compute the distance between two points in the plane very quickly, which means evaluating an expression of the form $\sqrt{\Delta x^2 + \Delta y^2}$. Finding the exact value takes too much time. Using the L_1 norm $|\Delta x| + |\Delta y|$ is not quite accurate enough. A good compromise is to define an approximate distance D_α as
$D_\alpha = \max(|\Delta x|, |\Delta y|) + \alpha \min(|\Delta x|, |\Delta y|)$, where $0 < \alpha < 1$ is a fraction with small numerator and denominator.

Values of α that are commonly used are 1/2 and 1/3. The following graph shows the sets $D_{1/2} = 5$ (the inner polygon), $D_{1/3} = 5$ (the outer polygon) and r = 5 (the circle).



Note that D_α is symmetric with respect to the Δx and Δy axes, and the lines $\Delta y = \pm\Delta x$. It's amusing to note that $D_{1/3}(4,3) = 4 + (1/3)\cdot 3 = 5$. That is, this approximation is perfect for the 3,4,5 right triangle.

What is the best value of α? That is, what value of α will make the octagon closest to the circle?

It makes sense to use least squares and seek to minimize the square of the difference between the approximation and the exact value as we go around a circle. For convenience, we can scale so that the radius of the circle is 1 and because of symmetry we only need to consider $0 \le \theta \le \pi/4$. Let *m* be the slope of the line segment in this range of *θ*. Then $m = -\frac{1}{\alpha}$. Since $0 < \alpha < 1, m < -1$.

The error is $\varepsilon(m) = \int_0^{\pi/4} (f(m,\theta)-1)^2 d\theta$, where $f(m,\theta) = \frac{m}{m\cos\theta - \sin\theta}$. To minimize the error first note that as $m \to -\infty$, $\alpha \to 0$ and D_α approaches a circumscribed square, so $\lim_{m\to-\infty} \varepsilon(m) = \int_0^{\pi/4} \left(\tan^{-1}(\theta)-1\right)^2 d\theta = \int_0^1 \left(u-1\right)^2 \sec^2 u du > \int_0^1 (u-1)^2 du = 1/3$. Then, using Maple, we find that $\varepsilon(-1) \approx 0.0389$ and $\varepsilon'(m) = 0 \Rightarrow m \approx -3.232$ with $f(-3.232) \approx 0.00098$. So the minimum error corresponds to $\alpha \approx \frac{-1}{-3.232} \approx .309$, and α = 1/3 appears to be a good choice.

III Non-routine problems for high school students, using the Pythagorean Theorem:

Problem 1. Let P be any point inside rectangle ABCD. Show that $AP^2 + PC^2 = BP^2 + PD^2$.

Hint if necessary: Draw two line segments through P, one parallel to each pair of sides of the rectangle, and use the Pythagorean theorem on the right triangles formed.

Students could be guided to discover this result using Geometer's Sketchpad.

Extra questions: Does the point really have to be inside the rectangle for the result to hold? Suppose we left out the exponents in the equation. Would the result still be true?

Problem 2. A two-mile length of railroad track has been constructed (1 mile = 5,280 feet). In hot weather the track might expand. To account for this, the track is hinged at either end and in the middle, to allow the track to expand into a very shallow isosceles triangle. On a very hot day, the track expands by two feet. Approximately how high off the ground is the track at the middle?

This works very well as an oral problem. Ask students to guess the answer before doing any computations. Most students will probably guess around a foot or so. Then have them apply the Pythagorean Theorem, to one of the right triangles formed with hypotenuse 5281 feet and leg 5280 feet. Without using a calculator or even paper and pencil the students ought to be able to work out that the height will be greater than $\sqrt{10,000}$, or 100 feet!