Mathematics for Computer Graphics

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  - Determinant of a Matrix / Matrix Inverse

Coordinate Reference Frames

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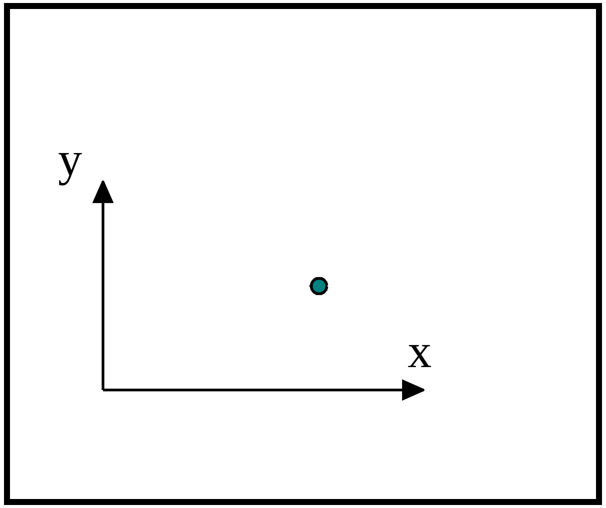
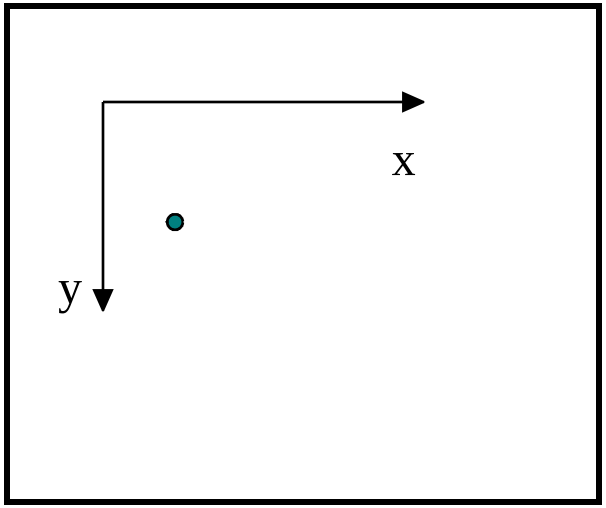
- **Coordinate Reference Frames**
  - Cartesian coordinate system
    - x, y, z 좌표축사용, 전형적 좌표계
  - Non-Cartesian coordinate system
    - 특수한 경우의 object표현에 사용.
    - Polar, Spherical, Cylindrical 좌표계 등

2D Cartesian Reference System

- **2D Cartesian Reference Frames**



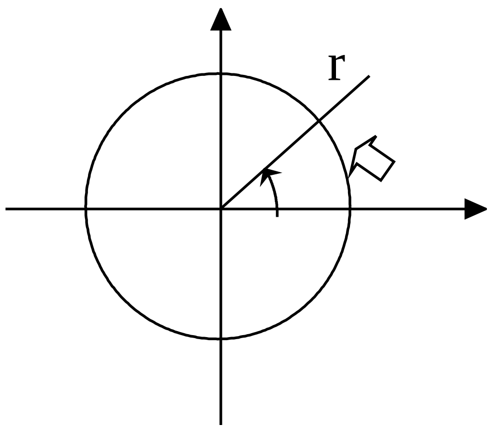
Coordinate origin at the lower-left screen corner



Coordinate origin in the upper-left screen corner

Polar Coordinates

- 가장 많이 쓰이는 **Non-Cartesian System**



$$x \quad r\cos \quad , \qquad y \quad r\sin$$

$$r \quad \sqrt{x^2 \quad y^2}, \qquad \tan^{1} \frac{y}{x}$$

$$s \quad r$$

- **Elliptical Coordinates, Hyperbolic or Parabolic Plane Coordinates** 등 원 이외에 **Symmetry**를 가진 다른 **2**차 곡선들로도 좌표계 표현 가능

Why Polar Coordinates?

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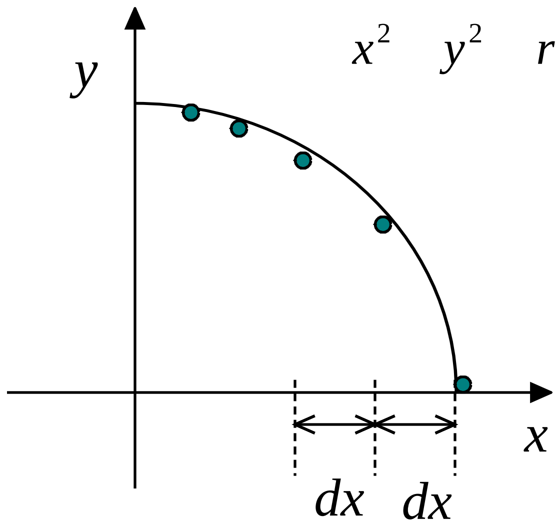
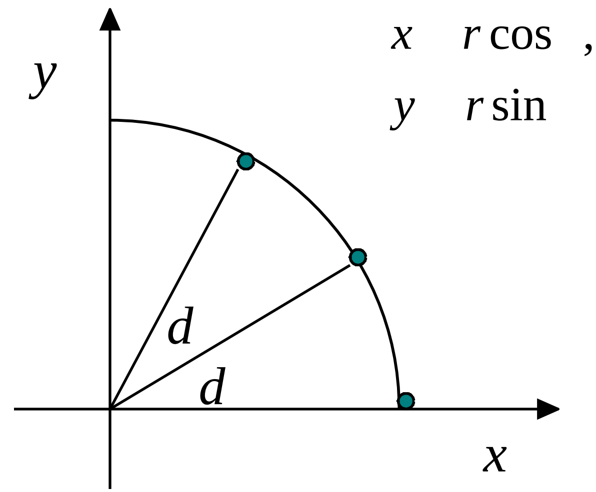
- **Circle**
  - 2D Cartesian : 비균등 분포

➔ Polar Coordinate



균등하게 분포되지 않은 점들

Cartesian Coordinates



연속된 점들 사이에 일정간격유지

Polar Coordinates

3D Cartesian Reference Frames

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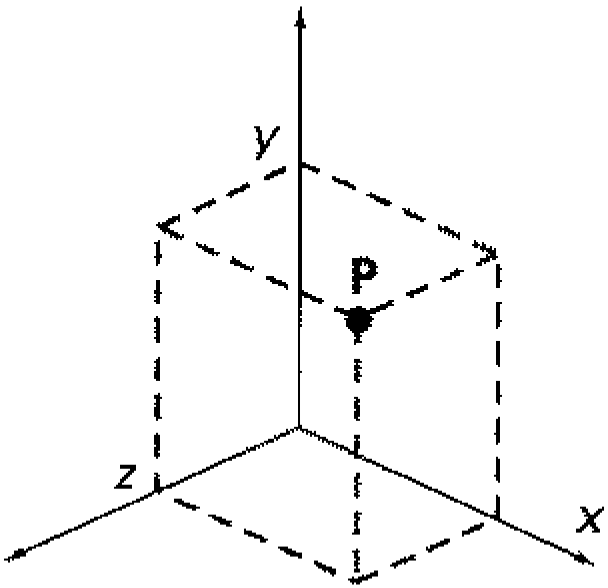

Three Dimensional Point

3D Cartesian Reference Frames

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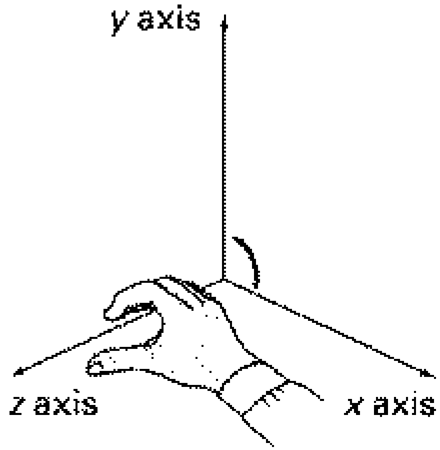

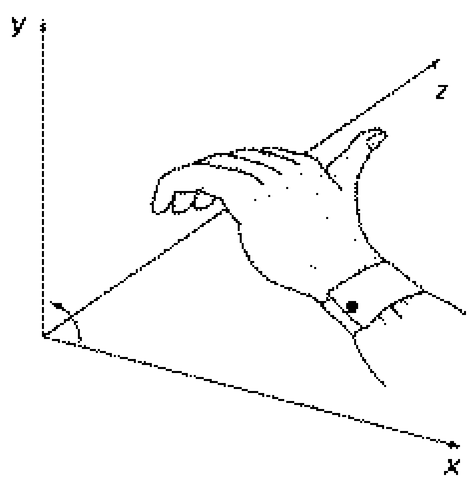

- 오른손 좌표계
  - 대부분의 Graphics Package에서 표준
- 왼손 좌표계
  - 관찰자로부터 얼마만큼 떨어져 있는지 나타내기에 편리함
  - Video Monitor의 좌표계

3D Curvilinear Coordinate Systems

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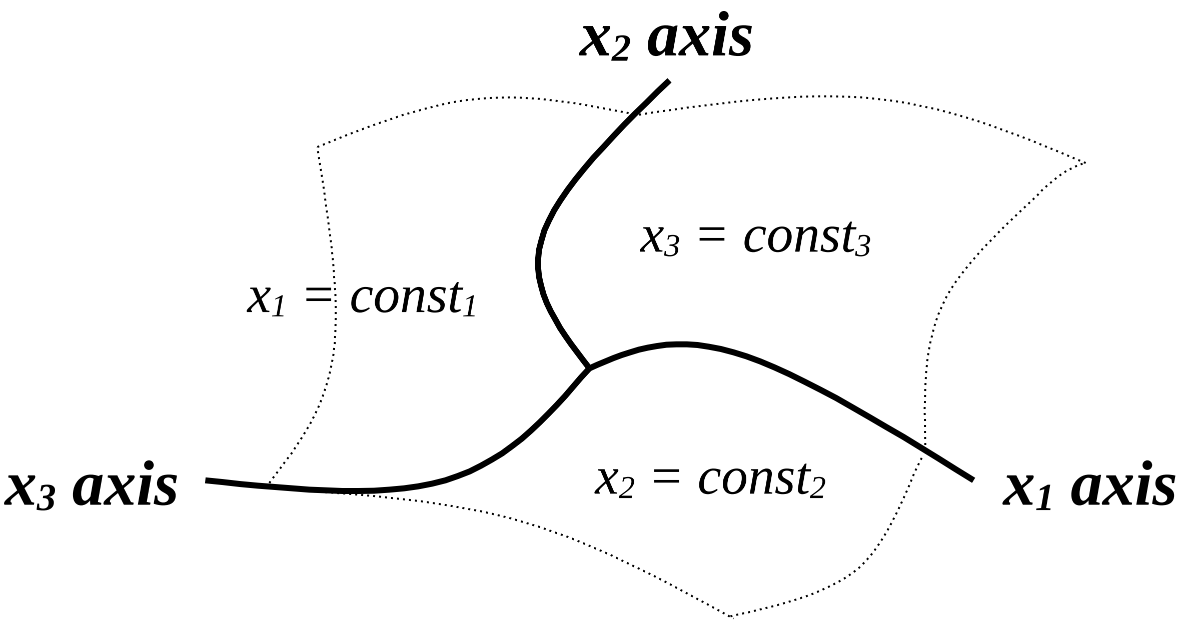
- **General Curvilinear Reference Frame**
  - Orthogonal coordinate system
    - Each coordinate surfaces intersects at right angles



A general Curvilinear coordinate reference frame

3D Non-Cartesian System

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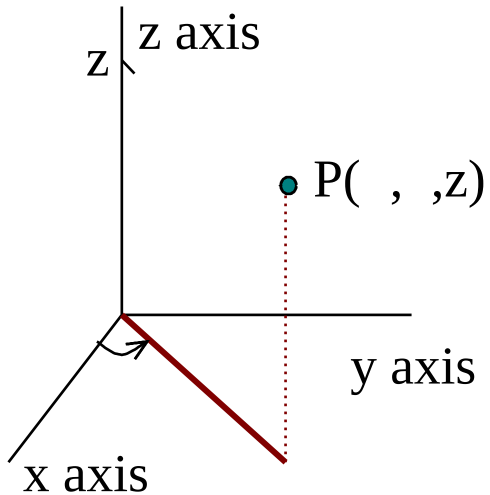
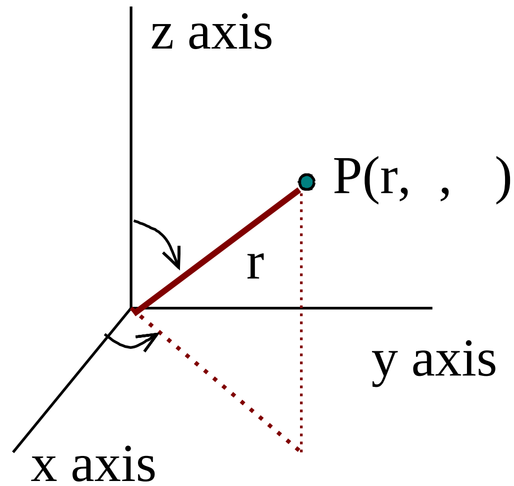
- **Cylindrical Coordinates**
- **Spherical Coordinates**





$$x \qquad \cos$$
$$y \qquad \sin$$
$$z \qquad z$$

$$x \qquad r\cos \qquad \sin$$
$$y \qquad r\sin \qquad \sin$$
$$z \qquad r\cos$$

Points and Vectors

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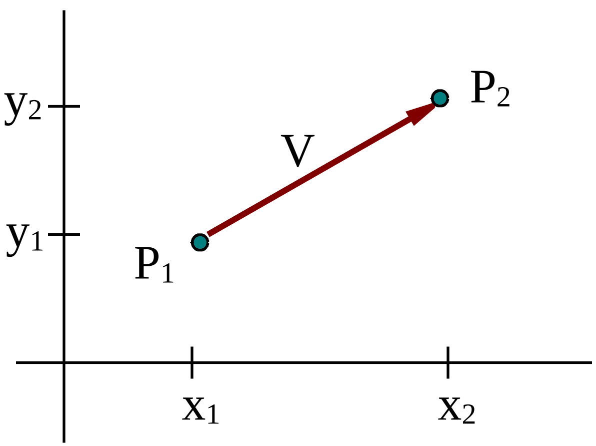
- **Point**: 좌표계의 한 점을 차지, 위치표시
- **Vector**: 두 **position**간의 차로 정의

$$V \quad P_2 \quad P_1 \quad (x_2 \quad x_1, y_2 \quad y_1) \quad (V_x, V_y)$$

  - Magnitude와 Direction으로도 표기



$$|V| \quad \sqrt{V_x^{\,2} \quad V_y^{\,2}}$$

$$\tan^{\ 1} \frac{V_y}{V_x}$$

Vectors

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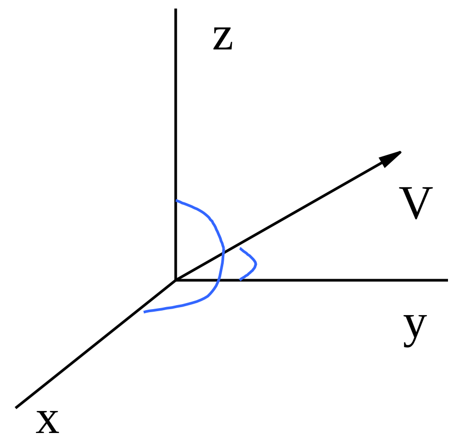
- **3차원에서의 Vector**

$$|V| \quad \sqrt{V_x^{\,2} \quad V_y^{\,2} \quad V_z^{\,2}}$$

$$\cos \quad \frac{V_x}{|V|}, \quad \cos \quad \frac{V_y}{|V|}, \quad \cos \quad \frac{V_z}{|V|}$$

$$\cos^2 \quad \cos^2 \quad \cos^2 \quad 1$$



- **Vector Addition and Scalar Multiplication**

$$V_1 \quad V_2 \quad (V_{1x} \quad V_{2x}, V_{1y} \quad V_{2y}, V_{1z} \quad V_{2z})$$

$$V \quad (\quad V_x, \quad V_y, \quad V_z)$$

Scalar Product

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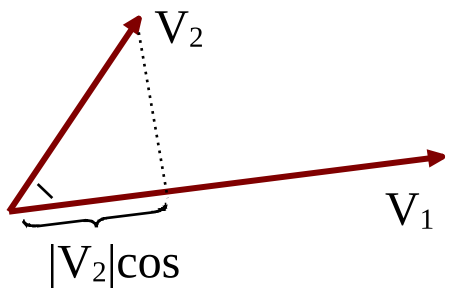
- **Definition**



$$V_1 \quad V_2 \quad |V_1||V_2|\cos \quad , \quad 0$$

Dot Product, Inner Product라고도 함

- **For Cartesian Reference Frame**

$$V_1 \quad V_2 \quad V_{1x}V_{2x} \quad V_{1y}V_{2y} \quad V_{1z}V_{2z}$$

- **Properties**
  - Commutative $V_1 \quad V_2 \quad V_2 \quad V_1$
  - Distributive $V_1 \quad (V_2 \quad V_3) \quad V_1 \quad V_2 \quad V_1 \quad V_3$

Vector Product

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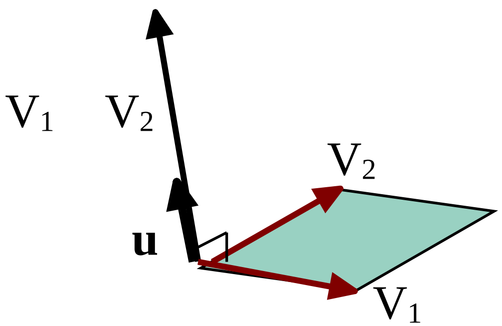
- **Definition**



$$V_1 \quad V_2 \quad \mathbf{u}\,|V_1|\,|V_2|\,\sin \quad , \quad 0$$

Cross Product, Outer Product라고도 함

- **For Cartesian Reference Frame**

$$V_1 \quad V_2 \quad (V_{1y}V_{2z} \quad V_{1z}V_{2y}, V_{1z}V_{2x} \quad V_{1x}V_{2z}, V_{1x}V_{2y} \quad V_{1y}V_{2x})$$

- **Properties**
  - AntiCommutative $V_1 \quad V_2 \quad (V_2 \quad V_1)$
  - Not Associative $V_1 \quad (V_2 \quad V_3) \quad (V_1 \quad V_2) \quad V_3$
  - Distributive $V_1 \quad (V_2 \quad V_3) \quad (V_1 \quad V_2) \quad (V_1 \quad V_3)$

Examples

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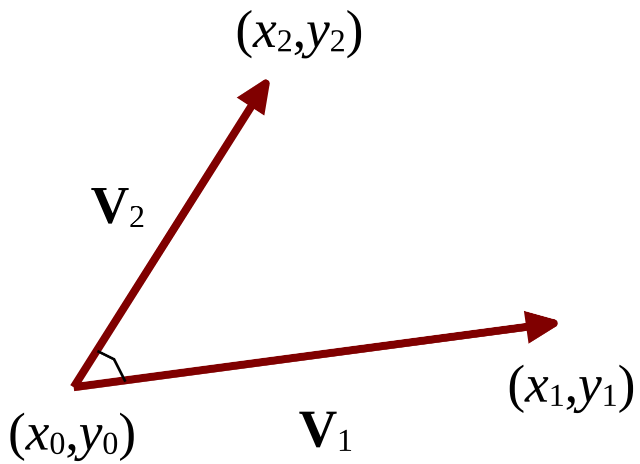
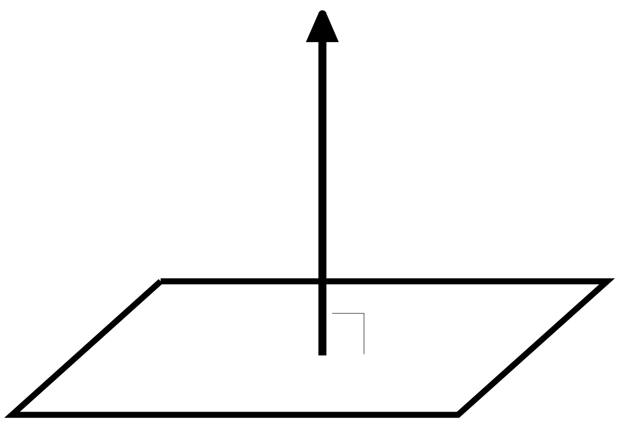
- **Scalar Product**
- **Vector Product**



Angle between Two Edges



Normal Vector of the Plane

Basis Vectors

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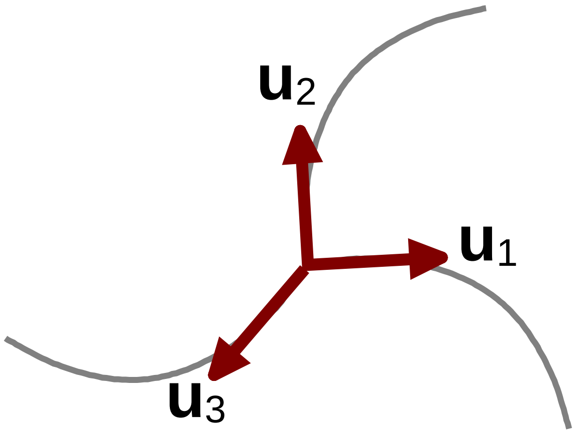
- **Basis** (or a Set of Base Vectors)
  - Specify the coordinate axes in any reference frame
  - Linearly independent set of vectors

  ➔ *Any other vector in that space can be written as linear combination of them*

- **Vector Space**
  - Contains scalars and vectors
  - *Dimension*: the number of base vectors



Curvilinear coordinate-axis vectors

Orthonormal Basis

- **Normal Basis + Orthogonal Basis**

$$\mathbf{u}_k \cdot \mathbf{u}_k = 1, \quad \text{for all } k$$

$$\mathbf{u}_j \cdot \mathbf{u}_k = 0, \quad \text{for all } j \neq k$$

- **Example**
  - Orthonormal basis for 2D Cartesian reference frame

$$\mathbf{u}_x = (1, 0) \quad \mathbf{u}_y = (0, 1)$$

  - Orthonormal basis for 3D Cartesian reference frame

$$\mathbf{u}_x = (1, 0, 0) \quad \mathbf{u}_y = (0, 1, 0) \quad \mathbf{u}_z = (0, 0, 1)$$

Metric Tensor

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- **Tensor**
  - Quantity having a number of components, depending on the tensor rank and the dimension of the space
  - Vector – tensor of rank 1, scalar – tensor of rank 0
- **Metric Tensor for any General Coordinate System**
  - Rank 2
  - Elements: $g_{jk} \quad \mathbf{u}_j \quad \mathbf{u}_k$
  - Symmetric: $g_{jk} \quad g_{kj}$

Properties of Metric Tensors

- **The Elements of a Metric Tensor can be used to Determine**
  - ***Distance*** between two points in that space
  - ***Transformation equations*** for conversion to another space
  - Components of various ***differential vector operators*** (such as gradient, divergence, and curl) within that space

Examples of Metric Tensors

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- **Cartesian Coordinate System**

$$\mathbf{u}_x \quad 1, 0 \qquad \mathbf{u}_y \quad 0, 1$$

$$g_{jk} \quad \begin{matrix} 1, & \text{if } j \quad k \\ 0, & \text{otherwise} \end{matrix}$$

- **Polar Coordinates**

$$\mathbf{u}_r \quad \mathbf{u}_x \cos \qquad \mathbf{u}_y \sin \quad ,$$

$$\mathbf{u} \qquad \mathbf{u}_x r \sin \qquad \mathbf{u}_y r \cos$$

$$\mathbf{g} \quad \begin{matrix} 1 & 0 \\ 0 & r^2 \end{matrix}$$

Matrices

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- **Definition**
  - A rectangular array of quantities

$$A \quad \begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{matrix}$$

- **Scalar Multiplication and Matrix Addition**

$$A \quad \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix} , \quad B \quad \begin{matrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{matrix}$$

$$A \quad B \quad \begin{matrix} a_{11} & b_{11} & a_{12} & b_{12} \\ a_{21} & b_{21} & a_{22} & b_{22} \end{matrix}$$

$$kA \quad \begin{matrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{matrix}$$

Matrix Multiplication

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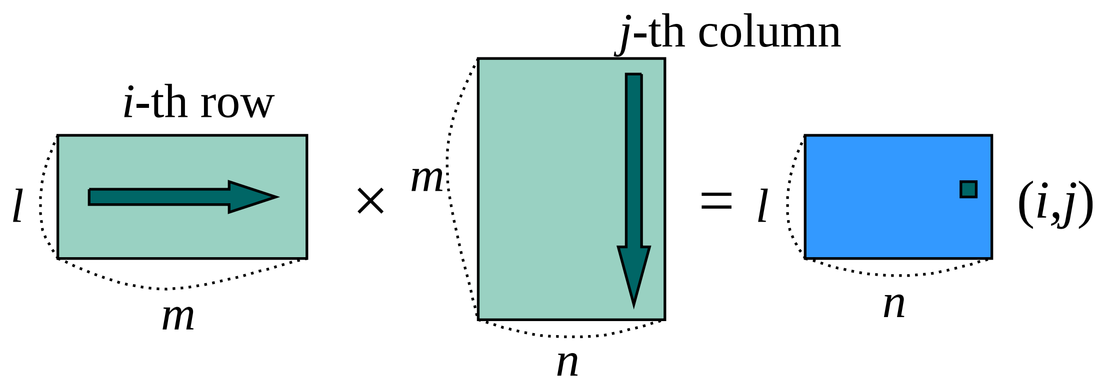
- **Definition**

$$C \quad AB$$

$$\Downarrow$$

$$c_{ij} \quad \overset{n}{\underset{k \quad 1}{}} \quad a_{ik} b_{kj}$$



- **Properties**
  - Not Commutative $\quad AB \qquad BA$
  - Associative $\quad (AB)C \qquad A(BC)$
  - Distributive $\quad A(B \qquad C) \qquad AB \qquad BC$
  - Scalar Multiplication $\quad (kA)B \qquad A(kB) \qquad k(AB)$

Matrix Transpose

- **Definition**
  - Interchanging rows and columns

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}, \quad \begin{bmatrix} a & b & c \end{bmatrix}^T = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

- **Transpose of Matrix Product**

$$(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T$$

Determinant of Matrix

- **Definition**
  - For a square matrix, combining the matrix elements to product a single number
- **2 × 2 matrix**

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

- **Determinant of n × n Matrix A (n ≥ 2)**

$$\det \mathbf{A} = \sum_{j=1}^{n} (-1)^{j+k} a_{jk} \det \mathbf{A}_{jk}$$

Inverse Matrix

- **Definition**

$$AA^{-1} = I \qquad A^{-1}A = I$$

  - Non-singular matrix
    - If and only if the determinant of the matrix is non-zero

- **2 × 2 matrix**

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- **Properties**

$$(A^{-1})^{-1} = A \qquad (AB)^{-1} = B^{-1}A^{-1} \qquad (A^T)^{-1} = (A^{-1})^T$$