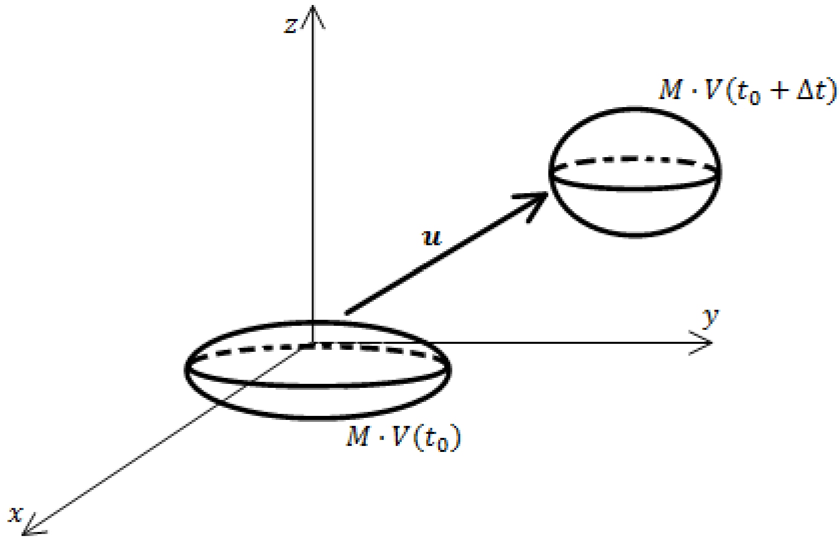
Derivation of continuity equation

A fundamental laws of Newtonian mechanics states the conservation of mass in an arbitrary material control



volume varying in time V_m.

A material volume contains the same portions of a fluid at all times. It may be defined by a closed bounding surface $S_m$eneveloping a portion of a fluid at certain time. Fluid elements cannot enter or leave this control volume. The movement of every point on the surface $S_m$is defined by the local velocity $\boldsymbol{u}$. So one can define:

$$0 = \frac{dM}{dt} = \frac{d}{dt} \int_{V_m} \rho \, dV.$$

Applying the Reynolds transport theorem and divergence theorem one obtains:

$$0 = \frac{dM}{dt} = \int_{V_m} \frac{\partial \rho}{\partial t} dV + \int_{S_m} \rho \boldsymbol{u} \cdot \boldsymbol{n} \, dS = \int_{V_m} \left(\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) \right) dV$$

Since this relation is valid for an arbitrary volume V_m, the integrand must be zero. Note that now it can easily be assumed that the volume is a fixed control volume (where fluid particles can freely enter and leave the volume) by taking account of mass fluxes through the surface S_m.

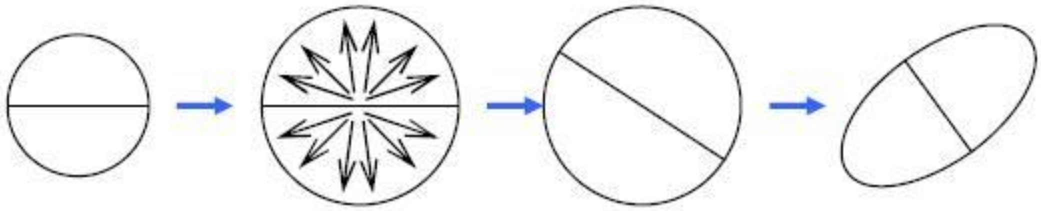
Thus

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) \qquad (1)$$

at all points of the fluid. For an incompressible fluid the change rate of density is zero. One can simplify (1) to:

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0.$$

Derivation of momentum equations



Expansion, rotation and deformation of a fluid parsel

forces and stresses

$$\mathbf{V} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w \tag{2}$$

$$\rho \frac{D\mathbf{V}}{Dt} = \rho\mathbf{F} + \mathbf{P} \tag{3}$$

where

$\mathbf{F}$- mass force per volume unit

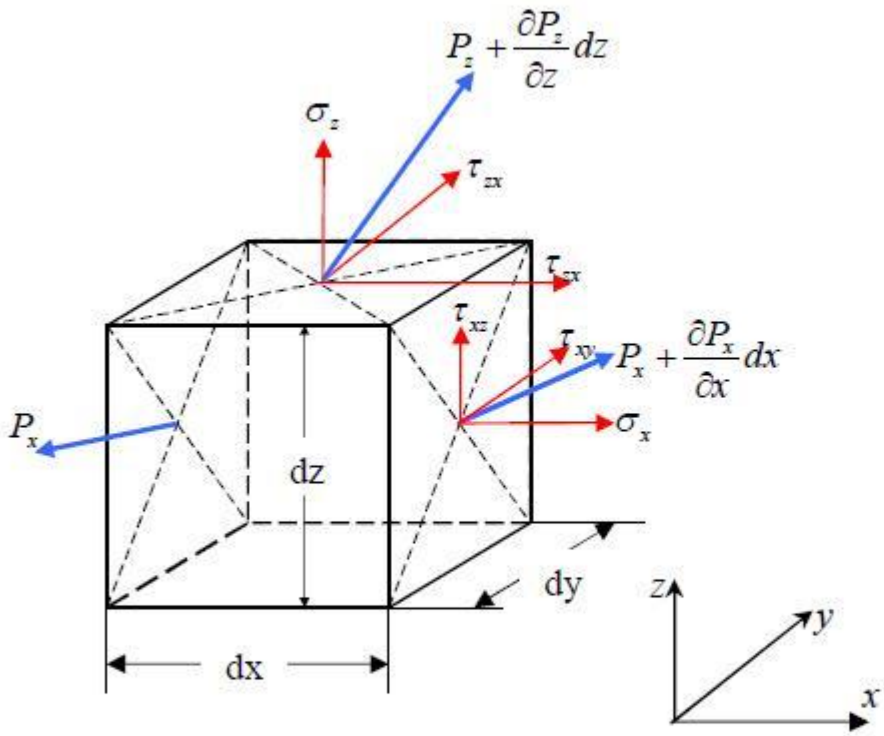
$\mathbf{P}$- surface force per volume unit

$$\mathbf{F} = \mathbf{i}F_x + \mathbf{j}F_y + \mathbf{k}F_z \tag{4}$$

There are two types of forces: body(mass) forces and surface forces. Body forces act on the entire control volume. The most common body force is that due to gravity. Electromagnetic phenomena may also create body forces, but this is a rather specialized situation.

Surface forces act on only surface of a control volume at a time and arise due to pressure or viscous stresses.

We find a general expression for the surface force per unit volume of a deformable body. Consider a rectangular parallelepiped with sides dx, dy, dz and hence with volume $dV = dxdydz$



At the moment we assume this parallelepiped isolated from the rest of the fluid flow , and consider the forces acting on the faces of the parallelepiped.

Let the left forward top of a parallelepiped lies in a point *O*

To both faces of the parallelepiped perpendicular to the axis x and having the area $dydz$ applied resulting stresses , equal to $\mathbf{p}_x$ and $\mathbf{p}_x + \frac{\partial \mathbf{p}_x}{\partial x}$ respectively

So we have

for x- direction $\frac{\partial \mathbf{p}_x}{\partial x} dxdydz$

for y- direction $\frac{\partial \mathbf{p}_y}{\partial y} dxdydz$

for z- direction $\frac{\partial \mathbf{p}_z}{\partial z} dxdydz$

$$\mathbf{P} = \frac{\partial \mathbf{p}_x}{\partial x} + \frac{\partial \mathbf{p}_y}{\partial y} + \frac{\partial \mathbf{p}_z}{\partial z} \tag{6}$$

$$\rho \frac{d\mathbf{V}}{dt} = \rho \mathbf{F} + \frac{\partial \mathbf{p}_x}{\partial x} + \frac{\partial \mathbf{p}_y}{\partial y} + \frac{\partial \mathbf{p}_z}{\partial z} \tag{7}$$

$$\left.\begin{aligned} \rho \tfrac{du}{dt} &= \rho F_x + \tfrac{\partial p_{xx}}{\partial x} + \tfrac{\partial p_{yx}}{\partial y} + \tfrac{\partial p_{zx}}{\partial z} \\ \rho \tfrac{dv}{dt} &= \rho F_y + \tfrac{\partial p_{xy}}{\partial x} + \tfrac{\partial p_{yy}}{\partial y} + \tfrac{\partial p_{zy}}{\partial z} \\ \rho \tfrac{dw}{dt} &= \rho F_z + \tfrac{\partial p_{xz}}{\partial x} + \tfrac{\partial p_{yz}}{\partial y} + \tfrac{\partial p_{zz}}{\partial z} \end{aligned}\right\} \tag{8}$$

$$\left.\begin{aligned} \rho \left(\tfrac{\partial u}{\partial t} + u \tfrac{\partial u}{\partial x} + v \tfrac{\partial u}{\partial y} + w \tfrac{\partial u}{\partial z} \right) &= \rho F_x + \tfrac{\partial p_{xx}}{\partial x} + \tfrac{\partial p_{yx}}{\partial y} + \tfrac{\partial p_{zx}}{\partial z} \\ \rho \left(\tfrac{\partial u}{\partial t} + u \tfrac{\partial u}{\partial x} + v \tfrac{\partial u}{\partial y} + w \tfrac{\partial u}{\partial z} \right) &= \rho F_y + \tfrac{\partial p_{xx}}{\partial x} + \tfrac{\partial p_{yx}}{\partial y} + \tfrac{\partial p_{zx}}{\partial z} \\ \rho \left(\tfrac{\partial u}{\partial t} + u \tfrac{\partial u}{\partial x} + v \tfrac{\partial u}{\partial y} + w \tfrac{\partial u}{\partial z} \right) &= \rho F_z + \tfrac{\partial p_{xx}}{\partial x} + \tfrac{\partial p_{yx}}{\partial y} + \tfrac{\partial p_{zx}}{\partial z} \end{aligned}\right\} \tag{9}$$

The force due to the stress is the product of the stress and the area over which it acts.

$$\mathbf{P}_x = \mathbf{i}\sigma_{xx} + \mathbf{j}\tau_{xy} + \mathbf{k}\tau_{xz} \tag{10}$$

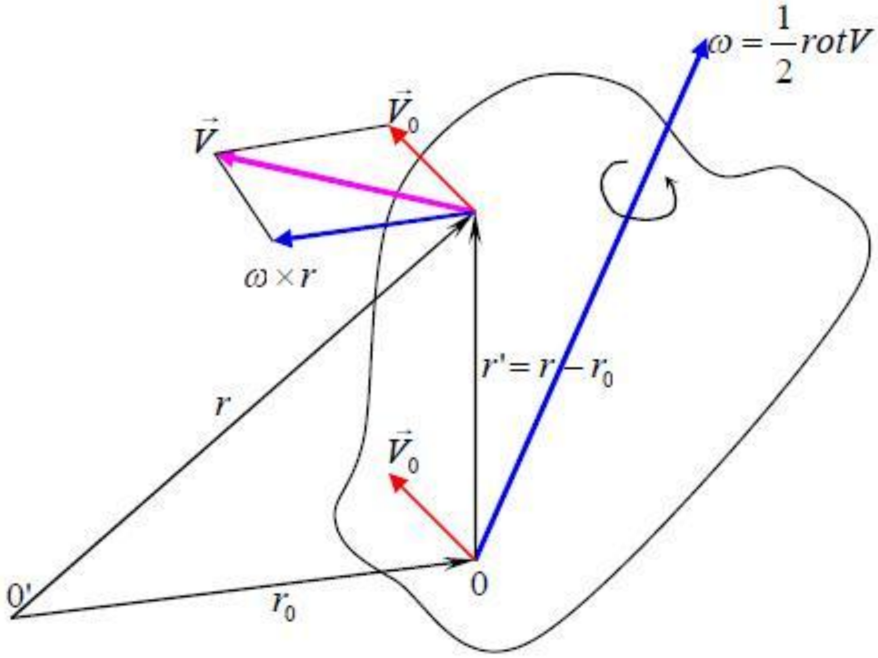
$$\mathbf{P}_y = \mathbf{i}\tau_{yx} + \mathbf{j}\sigma_{yy} + \mathbf{k}\tau_{yz} \tag{11}$$

$$\mathbf{P}_z = \mathbf{i}\tau_{zx} + \mathbf{j}\tau_{zy} + \mathbf{k} * \sigma_{zz} \tag{12}$$

$$\Pi = \begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix} \tag{13}$$

$$\left.\begin{aligned} \rho \tfrac{du}{dt} &= \rho F_x - \tfrac{\partial p}{\partial x} + \left(\tfrac{\partial \sigma'_x}{\partial x} + \tfrac{\partial \tau_{xy}}{\partial y} + \tfrac{\partial \tau_{xz}}{\partial z} \right) \\ \rho \tfrac{dv}{dt} &= \rho F_y - \tfrac{\partial p}{\partial y} + \left(\tfrac{\partial \tau_{xy}}{\partial x} + \tfrac{\partial \sigma'_y}{\partial y} + \tfrac{\partial \tau_{yz}}{\partial z} \right) \\ \rho \tfrac{dw}{dt} &= \rho F_z - \tfrac{\partial p}{\partial z} + \left(\tfrac{\partial \tau'_{xz}}{\partial x} + \tfrac{\partial \tau_{yz}}{\partial y} + \tfrac{\partial \sigma'_z}{\partial z} \right) \end{aligned}\right\} \tag{14}$$

deformation and rotation



$$\left.\begin{aligned} u &= u_0 + \left(\frac{\partial u}{\partial x}\right)_0 (x - x_0) + \left(\frac{\partial u}{\partial y}\right)_0 (y - y_0) + \left(\frac{\partial u}{\partial z}\right)_0 (z - z_0) \\ v &= v_0 + \left(\frac{\partial v}{\partial x}\right)_0 (x - x_0) + \left(\frac{\partial v}{\partial y}\right)_0 (y - y_0) + \left(\frac{\partial v}{\partial z}\right)_0 (z - z_0) \\ w &= w_0 + \left(\frac{\partial w}{\partial x}\right)_0 (x - x_0) + \left(\frac{\partial w}{\partial y}\right)_0 (y - y_0) + \left(\frac{\partial w}{\partial z}\right)_0 (z - z_0) \end{aligned}\right\} \quad (15)$$

$$\left.\begin{aligned} u &= u_0 + \omega_y (z - z_0) - \omega_z (y - y_0) \\ v &= v_0 + \omega_z (x - x_0) - \omega_x (z - z_0) \\ w &= w_0 + \omega_x (y - y_0) - \omega_y (x - x_0) \end{aligned}\right\} \quad (16)$$

$$\left.\begin{aligned} \omega_x &= \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \\ \omega_y &= \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \\ \omega_z &= \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \end{aligned}\right\} \quad (18)$$

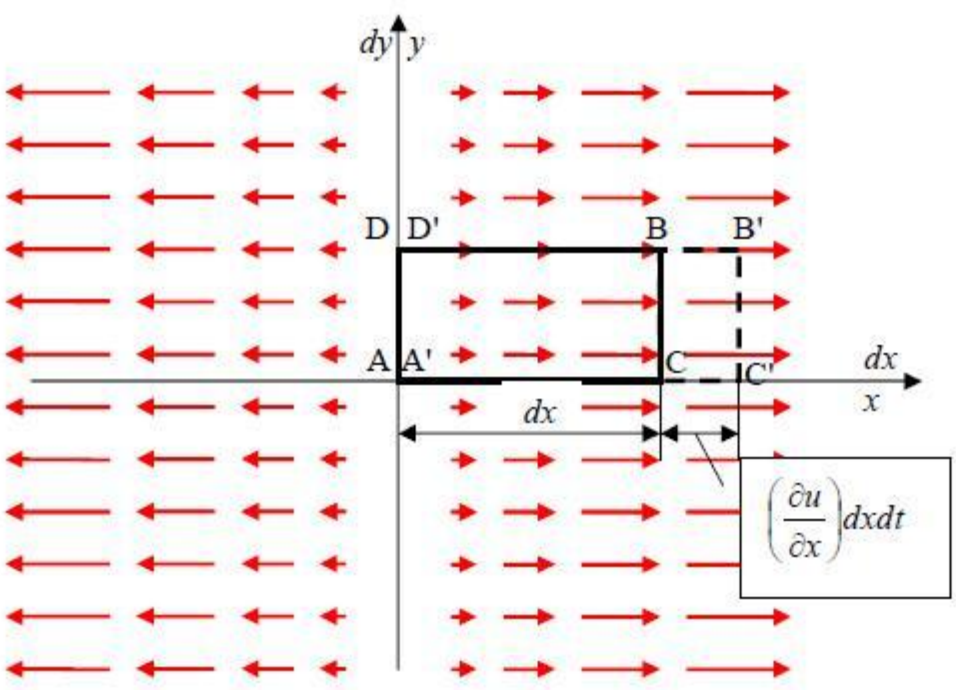
$$\left.\begin{aligned} u_{solid} &= u_0 + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)_0 (z - z_0) - \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)_0 (y - y_0) \\ v_{solid} &= v_0 + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)_0 (x - x_0) - \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)_0 (z - z_0) \\ w_{solid} &= w_0 + \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)_0 (y - y_0) - \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)_0 (x - x_0) \end{aligned}\right\} \quad (19)$$

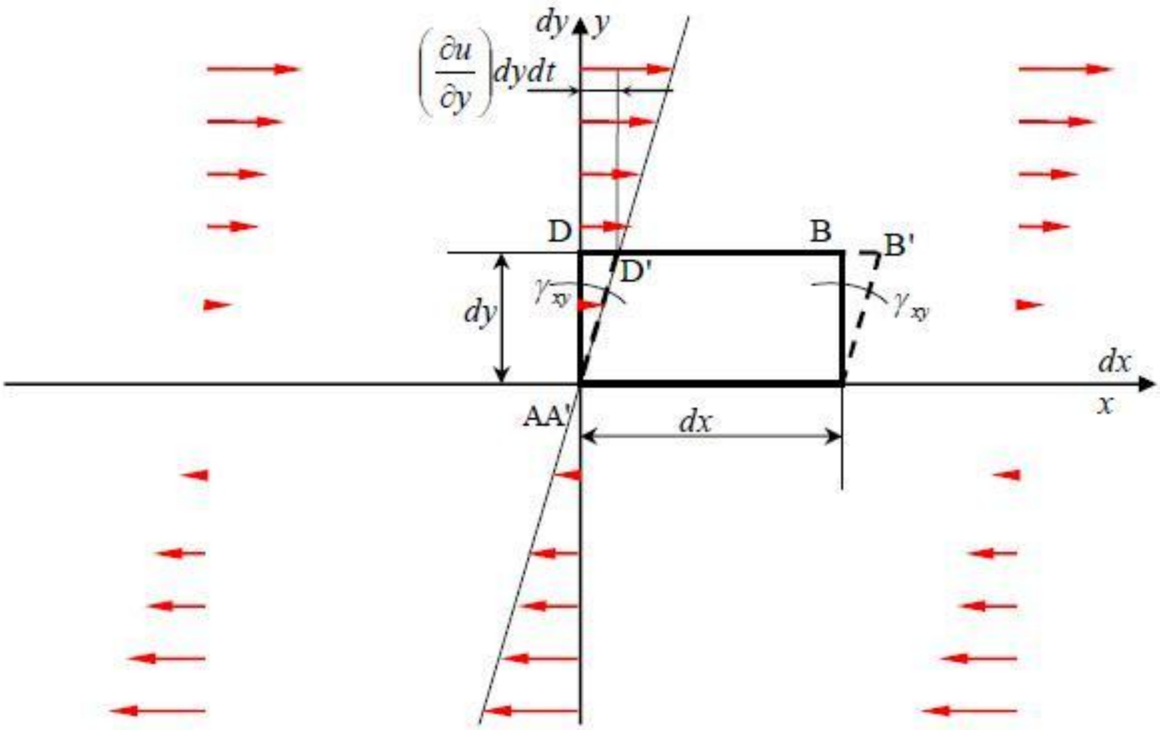
$$\left.\begin{aligned} u &= u_{solid} + u_{def} \\ v &= v_{solid} + v_{def} \\ w &= w_{solid} + w_{def} \end{aligned}\right\} \quad (20)$$

$$\left.\begin{aligned} u_{def} &= \left(\frac{\partial u}{\partial x}\right)_0 (x - x_0) + \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)_0 (y - y_0) + \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)_0 (z - z_0) \\ v_{def} &= \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)_0 (x - x_0) + \left(\frac{\partial v}{\partial y}\right)_0 (y - y_0) + \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)_0 (z - z_0) \\ w_{def} &= \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)_0 (x - x_0) + \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)_0 (y - y_0) + \left(\frac{\partial w}{\partial z}\right)_0 (z - z_0) \end{aligned}\right\} \quad (21)$$

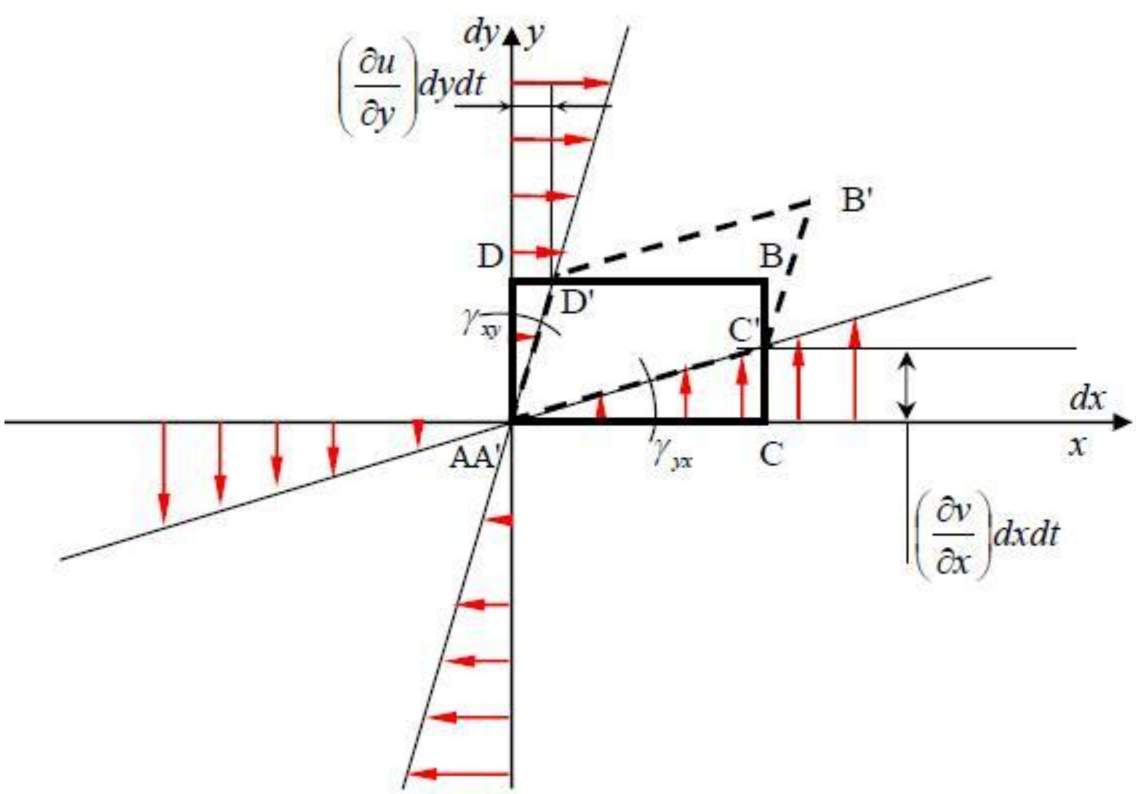
$$\dot{\epsilon}_{ij} \equiv \begin{pmatrix} \dot{\epsilon}_x & \dot{\epsilon}_{xy} & \dot{\epsilon}_{xz} \\ \dot{\epsilon}_{yx} & \dot{\epsilon}_y & \dot{\epsilon}_{yz} \\ \dot{\epsilon}_{zx} & \dot{\epsilon}_{zy} & \dot{\epsilon}_z \end{pmatrix} \equiv \quad (11)$$

$$\equiv \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{\partial u}{\partial x} \end{pmatrix} \quad (22)$$

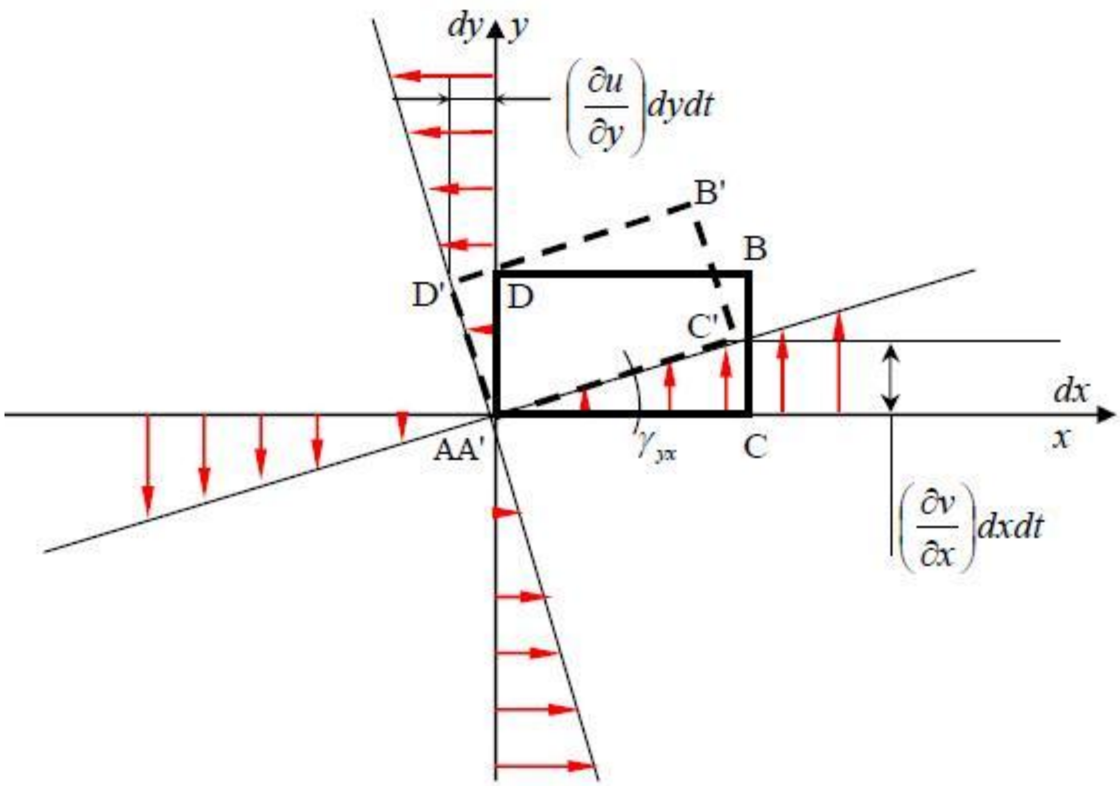




dy y
$\left(\frac{\partial u}{\partial y}\right)dydt$
D
B
B'
D'
γ_{xy}
dy
γ_{xy}
dx
x
AA'
dx

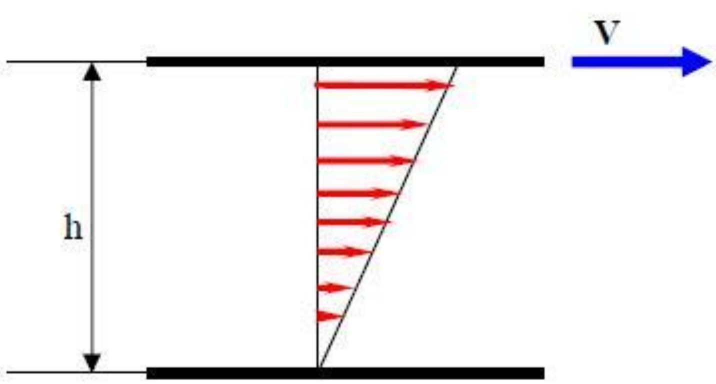


dy y
$\left(\frac{\partial u}{\partial y}\right)dydt$
B'
D
B
D'
γ_{xy}
C'
dx
x
AA'
γ_{yx}
C
$\left(\frac{\partial v}{\partial x}\right)dxdt$



Newtonian Fluids

Newton came up with the idea of requiring the stress τ to be linearly proportional to the time rate at which at which strain occurs. Specifically he studied the following problem. There are two flat plates separated by a distance h. The top plate is moved at a velocity V, while the bottom plate is held fixed.



Newton postulated (since then experimentally verified) that the shear force or shear stress needed to deform the fluid was linearly proportional to the velocity gradient:

$$\tau \propto \frac{V}{h} \qquad (2)$$

The proportionality factor turned out to be a constant at moderate temperatures, and was called the coefficient of viscosity, μ. Furthermore, for this particular case, the velocity profile is linear, giving $v/h = \partial u/\partial y$.

Therefore, Newton postulated:

$$\tau = \mu \frac{\partial u}{\partial y} \qquad (2)$$

Fluids that have a linear relationship between stress and strain rate are called *Newtonian fluids*. This is a property of the fluid, not the flow. Water and air are examples of Newtonian fluids, while blood is a non-Newtonian fluid.

Stokes Hypothesis

Stokes extended Newton's idea from simple 1-D flows (where only one component of velocity is present) to multidimensional flows. He developed the following relations, collectively known as *Stokes relations*

$$\sigma_x = 2\mu \frac{\partial u}{\partial x} + \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \qquad (12)$$

$$\sigma_y = 2\mu \frac{\partial v}{\partial y} + \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \qquad (12)$$

$$\sigma_z = 2\mu \frac{\partial w}{\partial z} + \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \qquad (12)$$

$$\tau_{xy} = \tau_{yx} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \qquad (12)$$

$$\tau_{xz} = \tau_{zx} = \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \qquad (12)$$

$$\tau_{zy} = \tau_{yz} = \mu \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) \qquad (12)$$

The quantity μ is called molecular viscosity, and is a function of temperature.

The coefficient λ was chosen by Stokes so that the sum of the normal stresses σ_x, σ_y and σ_z are zero.

Then

$$\lambda = -\frac{2}{3}\mu \qquad (12)$$

substitution

$$\begin{aligned}
\rho\frac{du}{dt} &= \rho F_x - \frac{\partial p}{\partial x} + 2\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] - \frac{2}{3}\frac{\partial}{\partial x}(\\
\rho\frac{dv}{dt} &= \rho F_y - \frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + 2\frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] - \frac{2}{3}\frac{\partial}{\partial y}(\\
\rho\frac{dw}{dt} &= \rho F_z - \frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] + 2\frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial z}\right) - \frac{2}{3}\frac{\partial}{\partial z}(
\end{aligned} \qquad (12)$$

Derivation of the energy equation

Equations

The instantaneous continuity equation (1), momentum equation (2) and energy equation (3) for a compressible fluid can be written as:

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_j}\left[\rho u_j\right] = 0 \qquad (1)$$

$$\frac{\partial}{\partial t}\left(\rho u_i\right) + \frac{\partial}{\partial x_j}\left[\rho u_i u_j + p\delta_{ij} - \tau_{ji}\right] = 0, \quad i = 1,2,3 \qquad (2)$$

$$\frac{\partial}{\partial t}\left(\rho e_0\right) + \frac{\partial}{\partial x_j}\left[\rho u_j e_0 + u_j p + q_j - u_i \tau_{ij}\right] = 0 \qquad (3)$$

For a Newtonian fluid, assuming Stokes Law for mono-atomic gases, the viscous stress is given by:

$$\tau_{ij} = 2\mu S_{ij}^{*} \qquad (4)$$

Where the trace-less viscous strain-rate is defined by:

$$S_{ij}^{*} \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{1}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij} \qquad (5)$$

The heat-flux, q_j, is given by Fourier's law:

$$q_j = -\lambda\frac{\partial T}{\partial x_j} \equiv -C_p\frac{\mu}{Pr}\frac{\partial T}{\partial x_j} \qquad (6)$$

Where the laminar Prandtl number Pr is defined by:

$$Pr \equiv \frac{C_p \mu}{\lambda} \tag{7}$$

To close these equations it is also necessary to specify an equation of state. Assuming a calorically perfect gas the following relations are valid:

$$\gamma \equiv \frac{C_p}{C_v} \quad , \quad p = \rho R T \quad , \quad e = C_v T \quad , \quad C_p - C_v = R \tag{8}$$

Where γ, C_p, C_v and R are constant.

The total energy e_0 is defined by:

$$e_0 \equiv e + \frac{u_k u_k}{2} \tag{9}$$

Note that the corresponding expression (15) for Favre averaged turbulent flows contains an extra term related to the turbulent energy.

Equations (1)-(9), supplemented with gas data for γ, Pr, μ and perhaps R, form a closed set of partial differential equations, and need only be complemented with boundary conditions.

Boundary conditions

Existence and uniqueness

The existence and uniqueness of classical solutions of the 3-D Navier-Stokes equations is still an open mathematical problem and is one of the Clay Institute's Millenium Problems. In 2-D, existence and uniqueness of regular solutions for all time have been shown by Jean Leray in 1933. He also gave the theory for the existence of weak solutions in the 3-D case while uniqueness is still an open question.

However, recently, Prof. Penny Smith submitted a paper, Immortal Smooth Solution of the Three Space Dimensional Navier-Stokes System, which may provide a proof of the existence and uniqueness.(It has a serious flaw, so the author withdrew the paper)

References