Chapter 5.5:

Poisson Distribution, Poisson Approximation to Binomial

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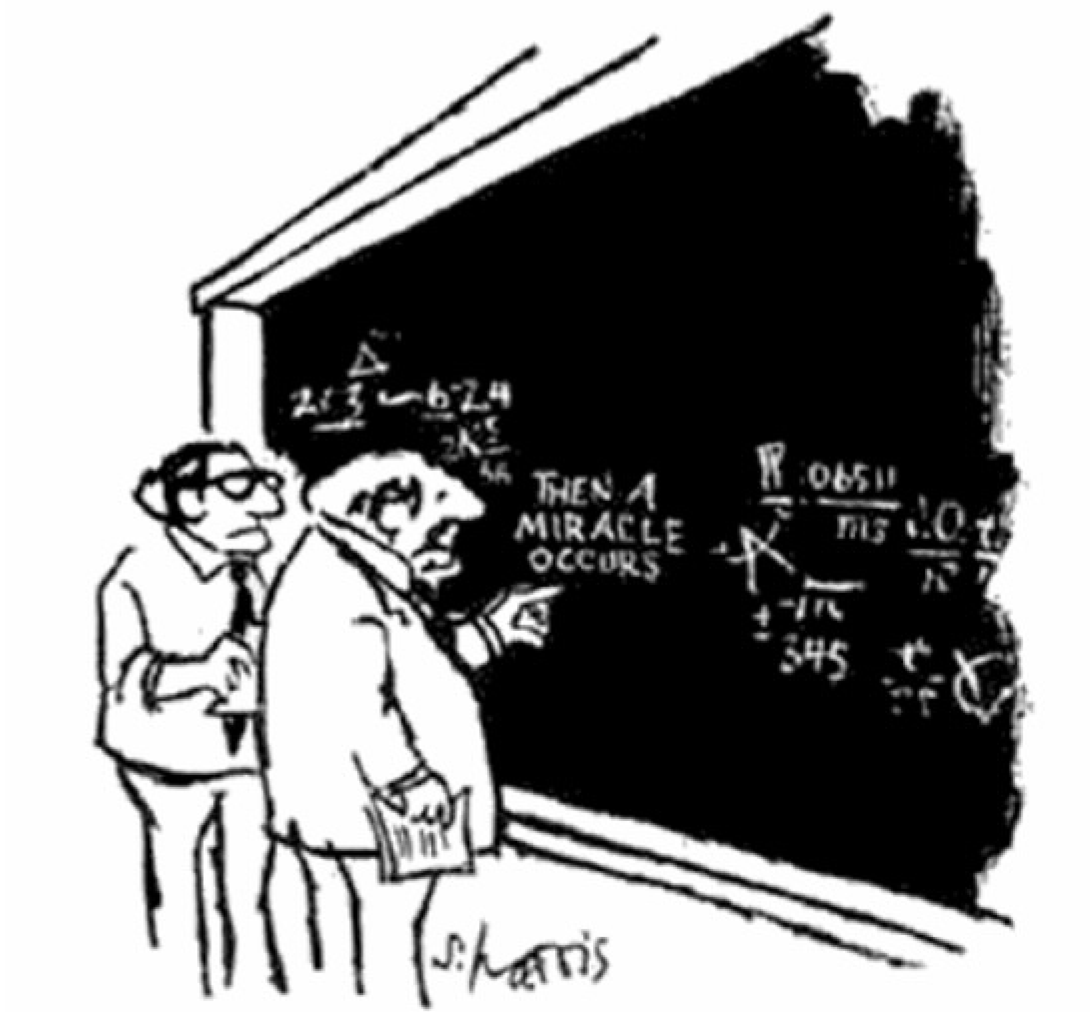
Lecture 14



"I THINK YOU SHOULD BE MORE EXPLICIT HERE IN STEP TWO."

Poisson Distribution

Sometimes we are interested in the number of rare events in a large interval. Let lambda (λ) be the average number of the rare events in this interval.

Such a random variable is called a Poisson random variable with parameter λ

Lambda is a "rate" or an average so if lambda is given over some period of "time", we might need to adjust it in context of the problem: *(examples)*

Poisson Distribution

Examples:

–The number of typos in a magazine.
–The number of tornados in Indiana.
–The number of people hit by lightning.

We must know an overall average of the event we expect to observe and the observed event must be countable.

Poisson Distribution

Notation: X ~ Poi(λ)

PMF:

$$p(x) \qquad \frac{{}^{x}e}{x!}$$

Expectation and Variance:

E(X) = λ
Var(X) = λ

Note: If $X_1 \sim Poi(\lambda_1)$ and $X_2 \sim Poi(\lambda_2)$ are independent, then $X_1 + X_2 \sim Poi(\lambda_1 + \lambda_2)$

Poisson Example #1

Earthquakes occur in the western United States with a rate of 2 per week (λ = 2). If we model the number of earthquakes as a Poisson random variable, what is the probability that there will be at least 3 earthquakes in a two-week period?

Let X be the number of earthquakes in a two-week period:

X ~ Poi(λ = 2 * two week period = 4)

$$p(x) \qquad \frac{4^{x} e^{\ 4}}{x!}$$

Poisson Example #1

Find the probability that there are at least 3 earthquakes in a two week period:

$$P(X \geq 3) = 1 - P(X < 3)$$

$$= 1 - [P(X=0) + P(X=1) + P(X=2)]$$

$$= 1 - \left(\frac{4^0 e^{-4}}{0!} + \frac{4^1 e^{-4}}{1!} + \frac{4^2 e^{-4}}{2!} \right)$$

$$= 1 - 13e^{-4} = 0.7619$$

Poisson Example #2a

The number of telephone calls coming into the central switchboard of an office building averages 4 per minute.

Let X be the number of phone calls in the next minute.

$$X \sim Poi(4)$$

Poisson Example #2a

Find the probability that no calls arrive during the next two minutes:

$$X \sim Poi(4*2\min 8)$$

$$P(X \; 0) \quad \frac{{}^{x}e}{x!}$$

$$\frac{8^{0}e^{8}}{0!}$$

$$e^{8} \quad 0.0003355$$

Poisson Example #2b

Find the mean and variance of the number of calls arriving during the next eight minutes:

$X \sim Poi(\lambda = 4 * 8 \text{ minutes} = 32)$, so $X \sim Poi(32)$

Then:

$$\mathbf{E(X) = Var(X) = \lambda = 32}$$

Poisson Example #2c

Each call costs 25 cents, plus there is an additional \$2 charge per hour just to keep the line open. Find the mean and standard deviation of the amount of money spent on telephone calls during the next hour.

λ = 4 calls/min * 60 min = 240....so: $X \sim Poi(240)$

And also let us define Y as the money spent on phone calls in the next hour such that:

$$Y = 0.25(X) + 2$$

Poisson Example #2c

$X \sim Poi(240)$ $Y = 0.25(X) + 2$

$E(Y) = E(0.25(X) + 2$
$= 0.25*E(X) + 2$
$= 0.25*240 + 2 = 62$

$Var(Y)=Var(0.25*X + 2)$
$=(0.25)^2*Var(X)$
$=(0.25)^2*240= 15$

$SD(Y) = sqrt(15) = 3.873$

Poisson Example #3

Let X ~ Poi(3) and Y ~ Poi(5) be independent.

What is P(X + Y) = 6?

X + Y ~ Poi(3 + 5 = 8)

P(X + Y = 6) $\frac{8^{68}e}{6!}$ 0.1221

Poisson Example #4

The current U.S. population consists of approximately 300 million people. Let us assume that 1 in 10 million people are struck by lightning in any given year and all of these strikes are independent of one another. Let X denote the number of people in the U.S. who were struck by lightning in a given year.

a) What is distribution of X?

b) What is the probability exactly 25 people will be struck by lightning in a given year?

c) How many people do we expect to be struck by lightning in a given year?

Poisson Example #4 (cont)

As it turns out if a random variable X is counting the number of "rare" occurrences in a large number of trials, and $n \geq 100$ and $p \leq .01$, then X follows an approximate Poisson distribution with $\lambda = np$. The Poisson distribution provides a good approximation to the binomial distribution when n is large and p is small, and also for rare not necessarily independent events in a large number of trials.

- What is the approximate probability exactly 25 people are struck by lightning in a given year?

- What is the approximate probability the between 24 and 26 (inclusive) people are struck by lightning in a given year?

Poisson Example #5

A manufacturer of Christmas tree light bulbs knows that 2% of its bulbs are defective. Let X denote the number of defective bulbs in a box of 200.

a) Assuming independence amongst the bulbs what is the distribution of X?

b) What is the probability there are exactly 4 defective bulbs in the box of 200 lights?

c) Using a Poisson approximation, calculate the approximate probability there are exactly 4 defective bulbs in the box of 200 lights?

Poisson Example #6

Flaws on a used computer tape occur on the average of one flaw per 1200 feet. Let X denote the number of flaws in a 4800-foot roll.

a) If we assume X follows a Poisson distribution, what is the corresponding value of lambda?

b) What is the probability the 4800-foot roll has at least one flaw?

Poisson Example #7

American Olympic Gold Medalist Lindsey Vonn has an average of 9 people asking for autographs per hour, all independent of one another while Apolo Ohno has an average of 12 people asking for autographs per hour, all independent of one another.

a) Lindsey takes a walk around Vancouver for 20 minutes. What is the probability she is asked at most 4 times for autographs during her walk?

b) Given Lindsey was asked at most 4 times for autographs in 20 minutes, what is the probability she is asked by exactly 3 people for autographs?

Poisson Example #8 (cont)

c) Every morning Apolo Ohno goes out for breakfast for 45 minutes. In one week, what is the probability 4 out of the 7 days he is asked by exactly 8 people for an autograph during his breakfast? (Assume each day is independent of all others)

d) All American Medalists are invited to a reception on the last day of the Olympics. Afterwards they spend 1 more hour in Vancouver before they go home. What is the probability that in that hour, Lindsey and Apolo are both asked by 10 people for an autograph?

Poisson Example #9

Customers arrive at a travel agency at a mean rate of 11 per hour. Assuming that the number of arrivals per hour has a Poisson distribution, give the probability that more than 10 customers arrive in a given hour? If the agency is open for 10 hours on any given day, what is the probability exactly 125 customers will arrive next Friday?

Poisson Example #9

If you buy a lottery ticket in 125 lotteries, in each of which your chance of winning a prize is 1/250, what is the approximate probability that you will win a prize:

a) At least once

b) Exactly once

c) At least twice