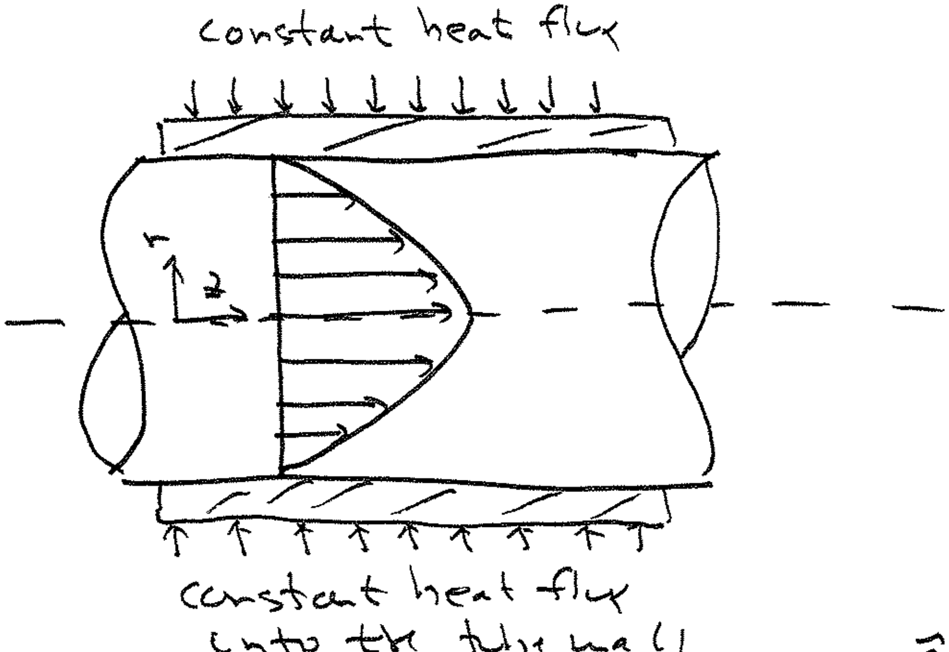
EXAMPLE 2

Nonisothermal Fluid Transport at a Constant Wall Heat Flux with a Laminar Velocity Profile



Laminar Flow Velocity Profile

$$v_z(r) = \frac{(\mathcal{P}_0 - \mathcal{P}_L)R^2}{4\mu L}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

$$= v_{z,max}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

where: $\mathcal{P}_0 = p_0 + \rho g L_0$; $\mathcal{P}_L = p_L + \rho g L$

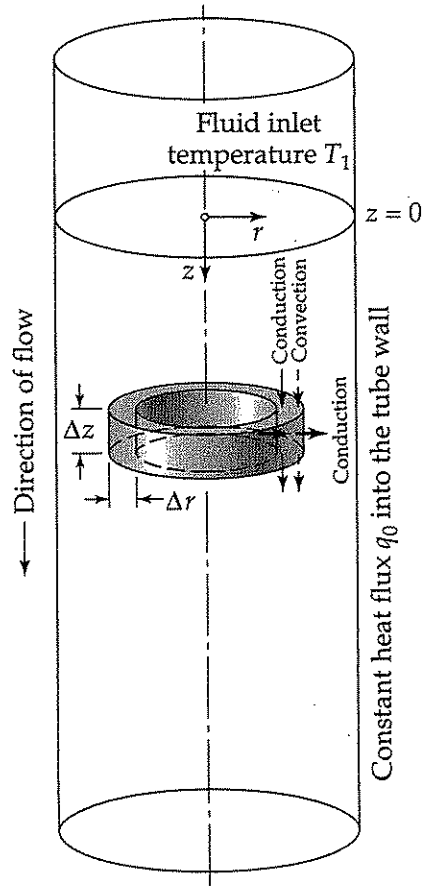
$$\rho \bar{C}_p v_{z,max}\left[1 - \left(\frac{r}{R}\right)^2\right]\frac{\partial T}{\partial z} = k\left[\frac{\partial^2 T}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right]$$

at $r = 0$, T is finite

at $r = R$, $k\frac{\partial T}{\partial r} = q_0$ (constant)

Nonisothermal Fluid Transport with a Laminar Velocity Profile - 1

Heating of a fluid in laminar flow through a circular tube showing the differential control volume



Assumptions

1. Steady-state $\Rightarrow T \neq f(t)$; $T = T(r, z)$
2. Constant fluid properties (ρ, C_p, μ, k ..)
3. Constant radial heat flux is applied at the wall $q_r = -q_0$. If the wall is heated, then q_0 = '+' and if chilled, then q_0 = '−'
4. Constant inlet temperature, i.e, $T = T_0$ at $z = 0$ & $0 \leq r \leq R$
5. The fluid velocity corresponds to a laminar flow profile $v_z(r) = v_{z,max}\left[1 - \left(\frac{r}{R}\right)^2\right]$
6. The variation in the velocity profile for $0 \leq r \leq R$ suggests that radial conduction of energy must be accounted for, in addition to axial conduction of energy
7. Fluid energy transport by convection occurs in the axial direction

Nonisothermal Fluid Transport with a Laminar Velocity Profile - 2

- The <u>average fluid velocity</u> is obtained by dividing the fluid volumetric flow rate by the cross-sectional area

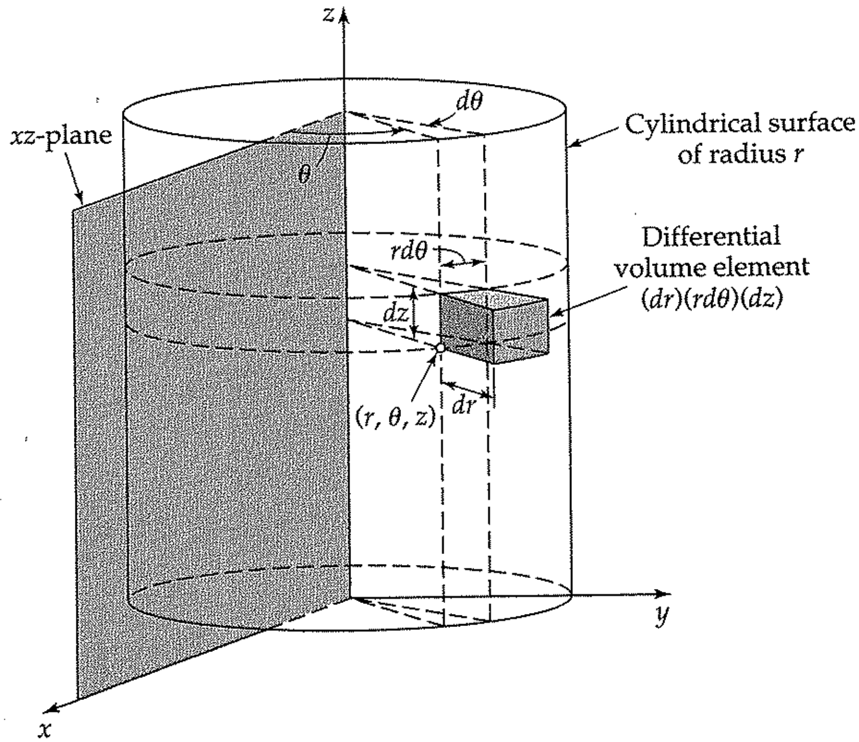
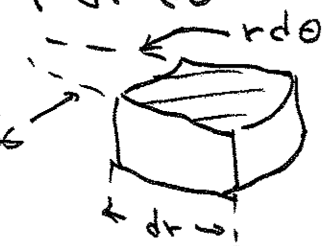


Fig. A.8-1. Differential volume element $r\,dr\,d\theta\,dz$ in cylindrical coordinates, and differential line elements dr, $r\,d\theta$, and dz. The differential surface elements are: $(r\,d\theta)(dz)$ perpendicular to the r direction (intermediate shading); $(dz)(dr)$ perpendicular to the θ direction (darkest shading); and $(dr)(r\,d\theta)$ perpendicular to the z direction (lightest shading).

- $v_z(r) = v_{z,max}\left[1 - \left(\frac{r}{R}\right)^2\right]$
- $dQ = v_z(r)\,dA$
- From the control volume, the differential area $\perp$ to the z-direction is

$$dA = r\,dr\,d\theta$$



- $dQ = v_z(r)\,r\,dr\,d\theta$

$$\langle v_z \rangle = \frac{\int dQ}{\int dA} = \frac{\int_0^{2\pi}\int_0^R v_z(r)\,r\,dr\,d\theta}{\int_0^{2\pi}\int_0^R r\,dr\,d\theta}$$

- Numerator = $Q = \int_0^{2\pi}\int_0^R v_{z,max}\left[1-\left(\frac{r}{R}\right)^2\right] r\,dr\,d\theta = v_{z,max}\int_0^{2\pi}\int_0^R \left(r - \frac{r^3}{R^2}\right) dr\,d\theta$

$$= v_{z,max}\int_0^{2\pi}\left[\frac{r^2}{2} - \frac{r^4}{4R^2}\right]_0^R d\theta = v_{z,max}\left[\frac{R^2}{2} - \frac{R^2}{4}\right]\int_0^{2\pi} d\theta = 2\pi\, v_{z,max}\frac{R^2}{4} = \frac{\pi}{2}R^2 v_{z,max}$$

Nonisothermal Fluid Transport with a Laminar Velocity Profile - 3

- Denominator = $A = \int_0^{2\pi}\int_0^R r\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^2}{2}\right]_0^R d\theta = \frac{2\pi R^2}{2} = \pi R^2$

$$\langle v_z \rangle = \frac{Q}{A} = \frac{\text{numerator}}{\text{Denominator}} = \frac{\frac{\pi}{2}R^2 v_{z,max}}{\pi R^2} = \frac{v_{z,max}}{2} \Rightarrow v_{z,max} = 2\langle v_z \rangle$$

- the laminar velocity profile is: $v_z(r) = v_{z,max}\left[1-\left(\frac{r}{R}\right)^2\right]$

$$= 2\langle v_z \rangle \left[1-\left(\frac{r}{R}\right)^2\right]$$

$$= 2 v_0 \left[1-\left(r/R\right)^2\right]$$

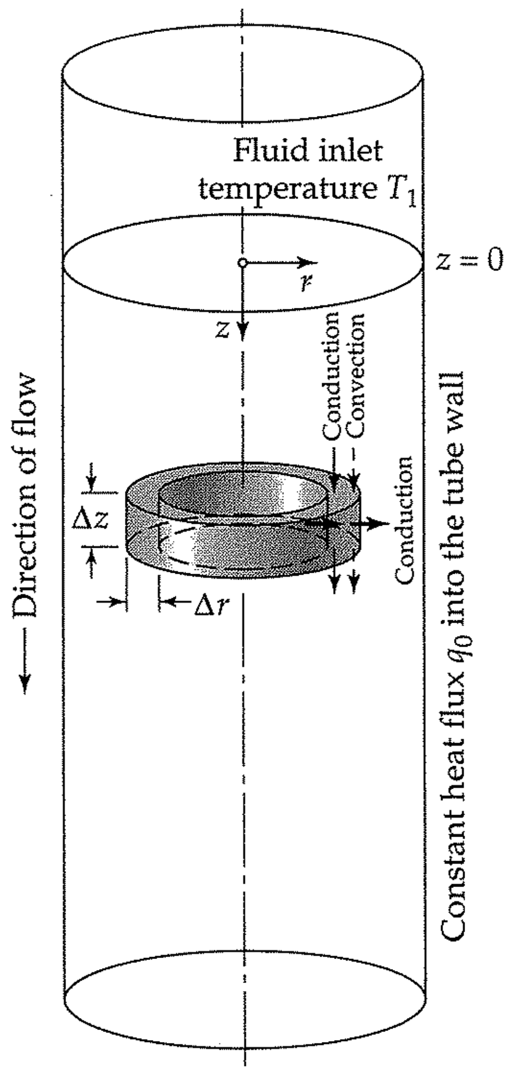
But: $v_{z,max} = 2\langle v_z \rangle = 2 v_0$

Text notation Eq 1.21

- The above velocity distribution $v_z(r)$ can be used in the energy balance where the velocity vector $\vec{v}$ is

$$\vec{v} = \hat{\delta}_r v_r + \hat{\delta}_\theta v_\theta + \hat{\delta}_z v_z \cong \hat{\delta}_z v_z(r,\theta,z) = \hat{\delta}_z v_z(r)$$

$v_z(r,\theta,z) = v_z(r)$



- Energy flux vector $= \vec{e} = \hat{\delta}_r e_r + \hat{\delta}_\theta e_\theta + \hat{\delta}_z e_z$

where $\vec{e} = \underbrace{\left(\frac{1}{2}\rho v^2 + \rho \hat{H}\right)\vec{v}}_{\text{convective}} + \underbrace{[\bar{\bar{\tau}} \cdot \vec{v}]}_{\text{work}} + \underbrace{\vec{q}}_{\text{conduction}}$

- Energy Balance

$$\begin{pmatrix}\text{Rate of energy input}\\ \text{by convection transport}\\ \text{\& molecular transport}\end{pmatrix} - \begin{pmatrix}\text{Rate of energy output}\\ \text{by convective transport}\\ \text{\& molecular transport}\end{pmatrix}$$

$$+ \begin{pmatrix}\text{Rate of work done}\\ \text{on system by}\\ \text{molecular transport}\end{pmatrix} - \begin{pmatrix}\text{Rate of work done}\\ \text{by system by}\\ \text{molecular transport}\end{pmatrix}$$

$$+ \begin{pmatrix}\text{Rate of work done}\\ \text{on system by external}\\ \text{forces}\end{pmatrix} + \begin{pmatrix}\text{Rate of energy}\\ \text{production}\end{pmatrix} = 0$$

At Steady State

- Total energy in at r $\quad e_r|_r \cdot 2\pi r \Delta z = (2\pi r e_r|_r)\Delta z$
- Total energy out at r $\quad e_r|_{r+\Delta r}\, 2\pi(r+\Delta r)\Delta z = (2\pi r e_r)|_{r+\Delta r}\Delta z$
- Total energy in at z $\quad e_z|_z \cdot 2\pi r \Delta r$
- Total energy out at $z+\Delta z$ $\quad e_z|_{z+\Delta z} \cdot 2\pi r \Delta r$

Nonisothermal Fluid Transport with a Laminar Velocity Profile - 5

- work done on fluid by gravity force $\qquad \rho g_z (2\pi r \Delta r \Delta z)\, v_z$

$1\,N = kg \cdot \frac{m}{s^2}$

$$\underbrace{\rho \frac{kg}{m^3}\, g_z \frac{m}{s^2}}_{\substack{N/m^3 \\ \text{(Force / volume)}}} \quad \underbrace{(2\pi r \Delta r \Delta z)\, m^3}_{\substack{m^3 \\ \text{(fluid volume)}}} \quad \underbrace{v_z\, m/s}_{\text{(fluid velocity)}}$$

- $$\underbrace{2\pi (r e_r)\big|_r \Delta z - 2\pi (r e_r)\big|_{r+\Delta r} \Delta z}_{\text{net energy in r-dirn}} + \underbrace{2\pi r \Delta r (e_z)\big|_z - 2\pi r \Delta r (e_z)\big|_{z+\Delta z}}_{\text{net energy in z-dirn}} + \underbrace{\rho g_z (2\pi r \Delta r \Delta z)\, v_z}_{\text{work done on fluid element}}$$

- Divide each term by the differential fluid volume element $2\pi r \Delta \Delta z$ & take the limit as Δr & $\Delta z \rightarrow 0$

$$\underbrace{\lim_{\Delta r \to 0} \frac{\left(\cancel{2\pi}(r e_r)\big|_r - 2\pi (r e_r)\big|_{r+\Delta r}\right)\cancel{\Delta z}}{\cancel{2\pi} r \Delta r \cancel{\Delta z}}}_{\text{cancel out } 2\pi \Delta z} + \underbrace{\lim_{\Delta z \to 0} \frac{\cancel{2\pi r \Delta r}(e_z)\big|_z - \cancel{2\pi r \Delta r}(e_z)\big|_{z+\Delta z}}{\cancel{2\pi r} \Delta r \Delta z}}_{\text{cancel } 2\pi r \Delta r} + \frac{\rho g_z (2\pi r \Delta r \Delta z)\, v_z(r)}{2\pi r \Delta r \Delta z}$$

$$= 0$$

- After canceling out like terms, the following eqn is obtained

$$-\lim_{\Delta r \to 0}\left\{\frac{(r e_r)\big|_{r+\Delta r} - (r e_r)\big|_r}{r\Delta r}\right\} - \lim_{\Delta z \to 0}\left\{\frac{(e_z)\big|_{z+\Delta z} - (e_z)\big|_z}{\Delta z}\right\} + \rho g_z v_z(r) = 0$$

- Applying the definition for the partial derivatives gives

$$-\frac{1}{r}\frac{\partial}{\partial r}(r e_r) - \frac{\partial e_z}{\partial z} + \rho g_z v_z(r) = 0$$

- Next, write out the components of the energy flux vector

$$\vec{e} = \hat{\delta}_r e_r + \hat{\delta}_\theta e_\theta + \hat{\delta}_z e_z = \hat{\delta}_r e_r + \hat{\delta}_z e_z \quad \text{since } e_\theta \equiv 0$$

$$= \left(\tfrac{1}{2}\rho v^2 + \rho\hat{H}\right)\vec{v} + [\overline{\overline{\tau}}\cdot\vec{v}] + \vec{q}$$

- To evaluate $[\overline{\overline{\tau}}\cdot\vec{v}]$ (dot product of the stress tensor $\overline{\overline{\tau}}$ & velocity vector $\vec{v}$), it can be shown that

$$[\overline{\overline{\tau}}\cdot\vec{v}] = \sum_i \hat{\delta}_i\left\{\sum_j \tau_{ij} v_j\right\}$$ where i & j are the coordinate directions

- In cylindrical coordinates:

$$[\overline{\overline{\tau}} \cdot \vec{v}] = \hat{\delta}_r \{ \tau_{rr} v_r + \tau_{r\theta} v_\theta + \tau_{rz} v_z \}$$

$$+ \hat{\delta}_\theta \{ \tau_{\theta r} v_r + \tau_{\theta\theta} v_\theta + \tau_{\theta z} v_z \}$$

$$+ \hat{\delta}_z \{ \tau_{zr} v_r + \tau_{z\theta} v_\theta + \tau_{zz} v_z \}$$

$[\overline{\overline{\tau}} \cdot \vec{v}]_r = \hat{\delta}_r \{ \tau_{rr} v_r + \tau_{r\theta} v_\theta + \tau_{rz} v_z \}$; r-component

$[\overline{\overline{\tau}} \cdot \vec{v}]_\theta = \hat{\delta}_\theta \{ \tau_{\theta r} v_r + \tau_{\theta\theta} v_\theta + \tau_{\theta z} v_z \}$; θ-component

$[\tau \cdot \vec{v}]_z = \hat{\delta}_z \{ \tau_{zr} v_r + \tau_{z\theta} v_\theta + \tau_{zz} v_z \}$; z-component

- Since $v_r = v_\theta = 0$ & $v_z = v_z(r)$,

$[\overline{\overline{\tau}} \cdot \vec{v}]_r = \hat{\delta}_r \tau_{rz} v_z$; $[\overline{\overline{\tau}} \cdot \vec{v}]_\theta = \hat{\delta}_\theta \tau_{\theta z} v_z$; $[\overline{\overline{\tau}} \cdot \vec{v}]_z = \hat{\delta}_z \tau_{zz} v_z$

Components of the Fluid Stress Tensor $\overline{\overline{\tau}}$ for Newtonian Fluids in Cartesian Coordinates (x, y, z)

$$[\boldsymbol{\tau} = -\mu(\nabla \mathbf{v} + (\nabla \mathbf{v})^\dagger) + (\tfrac{2}{3}\mu - \kappa)(\nabla \cdot \mathbf{v})\boldsymbol{\delta}]$$

Cartesian coordinates (x, y, z):

$$\tau_{xx} = -\mu\left[2\frac{\partial v_x}{\partial x}\right] + (\tfrac{2}{3}\mu - \kappa)(\nabla \cdot \mathbf{v}) \qquad \text{(B.1-1)}^a$$

$$\tau_{yy} = -\mu\left[2\frac{\partial v_y}{\partial y}\right] + (\tfrac{2}{3}\mu - \kappa)(\nabla \cdot \mathbf{v}) \qquad \text{(B.1-2)}^a$$

$$\tau_{zz} = -\mu\left[2\frac{\partial v_z}{\partial z}\right] + (\tfrac{2}{3}\mu - \kappa)(\nabla \cdot \mathbf{v}) \qquad \text{(B.1-3)}^a$$

$$\tau_{xy} = \tau_{yx} = -\mu\left[\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right] \qquad \text{(B.1-4)}$$

$$\tau_{yz} = \tau_{zy} = -\mu\left[\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right] \qquad \text{(B.1-5)}$$

$$\tau_{zx} = \tau_{xz} = -\mu\left[\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right] \qquad \text{(B.1-6)}$$

in which

$$(\nabla \cdot \mathbf{v}) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \qquad \text{(B.1-7)}$$

[a] When the fluid is assumed to have constant density, the term containing $(\nabla \cdot \mathbf{v})$ may be omitted. For monatomic gases at low density, the dilatational viscosity κ is zero.

Nonisothermal Fluid Transport with a Laminar Velocity Profile – 9

Components of the Fluid Stress Tensor $\overline{\overline{\tau}}$ for

CYLINDRICAL (r, θ, z)

Cylindrical coordinates (r, θ, z):

$$\tau_{rr} = -\mu\left[2\frac{\partial v_r}{\partial r}\right] + \left(\tfrac{2}{3}\mu - \kappa\right)(\nabla \cdot \mathbf{v})$$

$$\tau_{\theta\theta} = -\mu\left[2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)\right] + \left(\tfrac{2}{3}\mu - \kappa\right)(\nabla \cdot \mathbf{v})$$

$$\tau_{zz} = -\mu\left[2\frac{\partial v_z}{\partial z}\right] + \left(\tfrac{2}{3}\mu - \kappa\right)(\nabla \cdot \mathbf{v})$$

$$\tau_{r\theta} = \tau_{\theta r} = -\mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]$$

$$\tau_{\theta z} = \tau_{z\theta} = -\mu\left[\frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z}\right]$$

$$\tau_{zr} = \tau_{rz} = -\mu\left[\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right]$$

in which

$$(\nabla \cdot \mathbf{v}) = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}$$

SPHERICAL (r, θ, ϕ)

Spherical coordinates (r, θ, ϕ):

$$\tau_{rr} = -\mu\left[2\frac{\partial v_r}{\partial r}\right] + \left(\tfrac{2}{3}\mu - \kappa\right)(\nabla \cdot \mathbf{v})$$

$$\tau_{\theta\theta} = -\mu\left[2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)\right] + \left(\tfrac{2}{3}\mu - \kappa\right)(\nabla \cdot \mathbf{v})$$

$$\tau_{\phi\phi} = -\mu\left[2\left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r + v_\theta\cot\theta}{r}\right)\right] + \left(\tfrac{2}{3}\mu - \kappa\right)(\nabla \cdot \mathbf{v})$$

$$\tau_{r\theta} = \tau_{\theta r} = -\mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]$$

$$\tau_{\theta\phi} = \tau_{\phi\theta} = -\mu\left[\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right]$$

$$\tau_{\phi r} = \tau_{r\phi} = -\mu\left[\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right]$$

in which

$$(\nabla \cdot \mathbf{v}) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(v_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

- The term $\nabla \cdot \vec{v} = 0$ for constant density systems

Fourier's Law of Heat Conduction in Various Coordinates

$$\vec{q} = -k \nabla T$$

k = thermal conductivity of fluid or solid, W/m·K

Cartesian coordinates (x, y, z):

$$q_x = -k \frac{\partial T}{\partial x}$$

$$q_y = -k \frac{\partial T}{\partial y}$$

$$q_z = -k \frac{\partial T}{\partial z}$$

Cylindrical coordinates (r, θ, z):

$$q_r = -k \frac{\partial T}{\partial r}$$

$$q_\theta = -k \frac{1}{r} \frac{\partial T}{\partial \theta}$$

$$q_z = -k \frac{\partial T}{\partial z}$$

Spherical coordinates (r, θ, ϕ):

$$q_r = -k \frac{\partial T}{\partial r}$$

$$q_\theta = -k \frac{1}{r} \frac{\partial T}{\partial \theta}$$

$$q_\phi = -k \frac{1}{r \sin \theta} \frac{\partial T}{\partial \phi}$$

[a] For mixtures, the term $\Sigma_\alpha (\overline{H}_\alpha / M_\alpha) \mathbf{j}_\alpha$ must be added to $-k\nabla T$ (see Eq. 19.3-3).

Nonisothermal Fluid Transport with a Laminar Velocity Profile - II

- Return to the energy balance equation

$$-\frac{1}{r}\frac{\partial}{\partial r}(r e_r) - \frac{\partial e_z}{\partial z} + \rho g_z v_z(r) = 0$$

- Energy flux vector: $\vec{e} = \left(\frac{1}{2}\rho v^2 + \rho \hat{H}\right)\vec{v} + [\overline{\overline{\tau}}\cdot\vec{v}] + \vec{q}$

$$= \hat{\delta}_r e_r + \hat{\delta}_\theta e_\theta + \hat{\delta}_z e_z$$

$$\vec{v} = \hat{\delta}_r v_r + \hat{\delta}_\theta v_\theta + \hat{\delta}_z v_z = \hat{\delta}_z v_z \quad \left\{ \begin{array}{l} \text{since} \\ v_r = v_\theta = 0 \end{array} \right.$$

- Components of $\vec{e} = [e_r \;\; e_\theta \;\; e_z]^T$; Determine one at-a-time

$$e_r = \left(\frac{1}{2}\rho v^2 + \rho \hat{H}\right)\underbrace{\cancel{v_r}}_{\to 0} + \underbrace{[\overline{\overline{\tau}}\cdot\vec{v}]_r + q_r}_{\tau_{rz} v_z - k\frac{\partial T}{\partial r}} \; ; \; v_r = 0 \;\&\; [\overline{\overline{\tau}}\cdot\vec{v}]_r = \tau_{rz} v_z$$

$$q_r = -k\,\partial T/\partial r$$

$$\tau_{rz} = \tau_{zr} = -\mu\left[\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right] = -\mu\frac{dv_z}{dr} \quad \text{since } v_r = 0\text{; substitute into above expression}$$

$$\boxed{e_r = -\left(\mu\frac{dv_z}{dr}\right)v_z - k\frac{\partial T}{\partial r}} \quad \left\{ \begin{array}{l} \text{final expression} \\ \text{for } e_r \end{array} \right.$$

- We also need $e_z \Rightarrow$ z-component of $\vec{e}$

$$e_z = \left(\frac{1}{2}\rho v^2 + \rho\hat{H}\right) v_z(r) + [\overline{\overline{\tau}}\cdot\vec{v}]_z + q_z$$

$$v^2 = v_r^2 + v_\theta^2 + v_z^2 = v_z^2 \text{ since } v_r = v_\theta = 0$$

$$[\overline{\overline{\tau}}\cdot\vec{v}]_z = \tau_{zz} v_z \; ; \; \tau_{zz} = -\mu\left[2\frac{\partial v_z}{\partial z}\right] + \left(\frac{2}{3}\mu - \kappa\right)\nabla\cdot\vec{v}$$ { From page 9 of notes

- Since $v_z = v_z(r)$, then $\partial v_z/\partial z = 0$ & $\nabla\cdot\vec{v} = 0$ for ρ = constant

This means that $[\overline{\overline{\tau}}\cdot\vec{v}]_z \equiv 0$;

- $q_z = -k\dfrac{\partial T}{\partial z}$ { from Fourier's Law of conduction, page 10 of notes

- $$e_z = \left(\frac{1}{2}\rho v_z^2 + \rho\hat{H}\right) v_z + \tau_{zz} v_z + q_z$$

$$= \left(\frac{1}{2}\rho v_z^2\right) v_z + \rho\hat{H} v_z - \left(2\mu\frac{\partial v_z}{\partial z}\right) v_z - k\frac{\partial T}{\partial z} \; ; \; \frac{\partial v_z}{\partial z} \equiv 0 \text{ since } v_z = v_z(r)$$

$$\boxed{e_z = \left(\frac{1}{2}\rho v_z^2\right) v_z + \rho\hat{H} v_z - k\frac{\partial T}{\partial z}}$$

Final expression for e_z

Nonisothermal Fluid Transport with a Laminar Velocity Profile - 13

- Since; $-\frac{1}{r}\frac{\partial}{\partial r}\left(r e_r\right) - \frac{\partial e_z}{\partial z} + \rho g_z v_z(r) = 0$

$$e_r = -\left(\mu \frac{dv_z}{dr}\right)v_z - k\frac{\partial T}{\partial r}; \quad e_z = \left(\frac{1}{2}\rho v_z^2\right)v_z + \rho \hat{H} v_z - k\frac{\partial T}{\partial z}$$

- Term 1: $-\frac{1}{r}\frac{\partial}{\partial r}\left(r e_r\right) = \frac{1}{r}\frac{\partial}{\partial r}\left[\mu r \frac{dv_z}{dr} v_z + k r \frac{\partial T}{\partial r}\right]$

$$= \frac{\mu}{r}\left[r\left(\frac{dv_z}{dr}\right)^2\right] + v_z \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)$$

Term 2; $-\frac{\partial e_z}{\partial z} = -\frac{\partial}{\partial z}\left[\left(\frac{1}{2}\rho v_z^2\right)v_z + \rho \hat{H} v_z - k\frac{\partial T}{\partial z}\right]$

$$= -\frac{1}{2}\rho v_z^2 \underbrace{\cancel{\left(\frac{\partial v_z}{\partial z}\right)}}_{\to 0} - \rho v_z^2 \underbrace{\cancel{\frac{\partial v_z}{\partial z}}}_{\to 0} - \rho\frac{\partial}{\partial z}\left(\hat{H} v_z\right) + k\frac{\partial^2 T}{\partial z^2}$$

$$= -\rho v_z \frac{\partial \hat{H}}{\partial z} - \rho \hat{H}\frac{\partial v_z}{\partial z} + k\frac{\partial^2 T}{\partial z^2}; \quad \frac{\partial v_z}{\partial z} = 0 \text{ since } v_z = v_z(r)$$

$$= -\rho v_z \frac{\partial \hat{H}}{\partial z} + k\frac{\partial^2 T}{\partial z^2}$$

- We showed earlier that if $\hat{H} = \hat{H}(T, P)$, that $d\hat{H} \approx \hat{C}_p\, dT$

This means that $\dfrac{\partial \hat{H}}{\partial z} = \hat{C}_p \dfrac{\partial T}{\partial z}$ if $\hat{C}_p = \text{constant}$

- It follows that: $-\dfrac{\partial e_z}{\partial z} = -\rho v_z \hat{C}_p \dfrac{\partial T}{\partial z} + k \dfrac{\partial^2 T}{\partial z^2}$

- $-\dfrac{1}{r}\dfrac{\partial}{\partial r}(r e_r) - \dfrac{\partial e_z}{\partial z} + \rho g_z v_z = 0$ $\Big\{$ substitute the above expressions

$$\frac{\mu}{r}\left[r\left(\frac{dv_z}{dr}\right)^2\right] + v_z \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) - \rho v_z \hat{C}_p \frac{\partial T}{\partial z} + k\frac{\partial^2 T}{\partial z^2}$$
$$+ \rho g_z v_z(r) = 0$$

- Rearrange to:

$$\rho \hat{C}_p v_z \frac{\partial T}{\partial z} = k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2}\right] + \mu\left(\frac{\partial v_z}{\partial r}\right)^2 + v_z\left[\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \rho g_z\right]$$

where $v_z = v_z(r) = 2 v_0 \left[1 - \left(\dfrac{r}{R}\right)^2\right]$