CONTINUITY EQUATIONS - All The Equations Are Same

So far we have obtained four different types of continuity equations, each one a direct product of the flow model and conditions used in its derivation. Two of them are of integral types and the other two are of PDE types of them again two are in conservation forms and other two in non-conservation form. However, these four equations are not fundamentally different equations, but the same equation expressed in different forms. The underlying principle of all the four equations are the same: “mass is conserved”. Furthur, each can be derived from the other via simple mathematical manipulations, as we will see below.

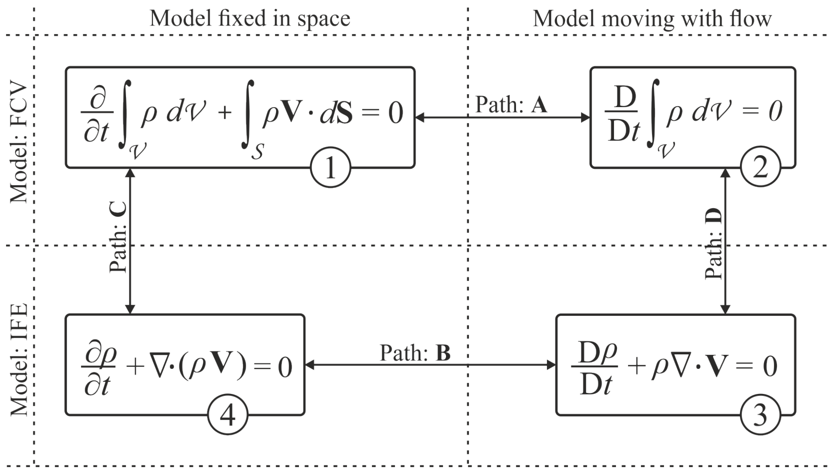

Figure 1: The four continuity equations

Path C

Let us examine path C and try to see if we can manipulate one equation to obtain the other. The equations involved are

$$\underbrace{\frac{\partial}{\partial t}\int_{\mathcal{V}} \rho \, d\mathcal{V} + \int_{\mathcal{S}} \rho \mathbf{V} \cdot d\mathbf{S} = 0}_{\text{Fixed FCV, Integral conservation form}} \quad \longleftrightarrow \quad \underbrace{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0}_{\text{Fixed IFE, PDE conservation form}} \tag{1}$$

Considering the LHS., we note that since the control volume is fixed in space, the limits of integration for the equation are constant and so the time derivative can be moved inside the integral, giving us

$$\int_{\mathcal{V}} \frac{\partial \rho}{\partial t} \, d\mathcal{V} + \int_{\mathcal{S}} \rho \mathbf{V} \cdot d\mathbf{S} = 0 \tag{2}$$

Using the divergence theorem, we can convert the surface integral into a volume integral

$$\int_{\mathcal{S}} \rho \mathbf{V} \cdot d\mathbf{S} = \int_{\mathcal{V}} \nabla \cdot (\rho \mathbf{V}) d\mathcal{V} \tag{3}$$

substituting into equation (2) we get

$$\int_{\mathcal{V}} \frac{\partial \rho}{\partial t} \, d\mathcal{V} + \int_{\mathcal{V}} \nabla \cdot (\rho \mathbf{V}) d\mathcal{V} = 0 \tag{4}$$

We further observe that the control volume being arbitrarily drawn in space, the only way for the integrals above to equal to zero is if the integrals are zero at every point within the control volume and hence can be rewritten as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0 \equiv \text{RHS} \tag{5}$$

Thus we have seen the similarities between the continuity equations obtained by using IFE and FCV fixed in space. We have converted a continuity equation in *Integral form* to one in *PDE form*.

Path B

In this path both the continuity equations are derived from IFE models resulting into PDE forms, but the equations are of two types, conservation and nonconservation form. We will see if it is possible to derive one from the other. Looking at the equations

$$\underbrace{\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{V})=0}_{\substack{\text{Fixed IFE, PDE}\\ \text{conservation form}}} \quad\longleftrightarrow\quad \underbrace{\frac{D\rho}{Dt}+\rho\mathbf{V}\cdot\nabla=0}_{\substack{\text{Moving IFE, PDE}\\ \text{nonconservation form}}} \tag{6}$$

Using the vector identity, derived by using the uv method from differential calculus

$$\nabla\cdot(\rho\mathbf{V})=(\rho\nabla\cdot\mathbf{V})+(\mathbf{V}\cdot\nabla\rho) \tag{7}$$

Applying the above identity to the LHS. of equation (6) we get

$$\begin{aligned} \frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{V}) &= 0\\ \Rightarrow\quad \underbrace{\frac{\partial\rho}{\partial t}+(\rho\nabla\cdot\mathbf{V})}_{\substack{\text{Substantial}\\ \text{derivative}}}+(\mathbf{V}\cdot\nabla\rho) &= 0\\ \Rightarrow\quad \frac{D\rho}{Dt}+(\mathbf{V}\cdot\nabla\rho) &= 0\equiv\text{RHS} \end{aligned} \tag{8}$$

In this case, we have successfully converted from conservation form to nonconservation form.

Path A

This path shows equations derived using FCV and thus are of Integral types. Once again we will see if we can relate between the conservation and nonconservation types. Looking at the equations

$$\underbrace{\frac{\partial}{\partial t}\int_{\mathcal{V}}\rho\,d\mathcal{V}+\int_{\mathcal{S}}\rho\mathbf{V}\cdot d\mathbf{S}=0}_{\substack{\text{Fixed FCV, Integral}\\ \text{conservation form}}} \quad\longleftrightarrow\quad \underbrace{\frac{D}{Dt}\int_{\mathcal{V}}\rho\,d\mathcal{V}=0}_{\substack{\text{Moving FCV, Integral}\\ \text{nonconservation form}}} \tag{9}$$

Looking closely at the RHS. equation $\frac{D}{Dt}\int_{\mathcal{V}}\rho\,d\mathcal{V}=0$, we observe the following points

- The volume integral is taken over the whole moving finite control volume $\mathcal{V}$ which is changing as it flows downstream
- The moving FCV is composed of an infinite number of infinitesimally small volumes of infinitesimally small mass
    - Each of these will have a volume $d\mathcal{V}$ which changes with flow
- Since the equation represents a substantial derivative (i.e., time rate of change of a moving element) the limits of this integral is represented by these same moving integrals, then the entire substantial derivative can be taken inside the integral

$$\frac{D}{Dt}\int_{\mathcal{V}}\rho\,d\mathcal{V}=\int_{\mathcal{V}}\frac{D(\rho\,d\mathcal{V})}{Dt}=0 \tag{10}$$

- Here $d\mathcal{V}$ represents an infinitesimally small element's volume, which itself is changing with time, i.e., both ρ and $d\mathcal{V}$ are variables: hence, this is a case of simple uv method from differential calculus

$$\int_{\mathcal{V}} \frac{D(\rho\, d\mathcal{V})}{Dt} \equiv \int_{\mathcal{V}} \frac{D\rho}{Dt}\, d\mathcal{V} + \int_{\mathcal{V}} \rho\, \frac{D(d\mathcal{V})}{Dt} = 0 \tag{11}$$

$$\text{Multiplying the last term by } d\mathcal{V} \text{ gives:} \int_{\mathcal{V}} \rho \left\{ \frac{D(d\mathcal{V})}{Dt} \right\} \mathrm{d}\mathcal{V} \tag{12}$$

In the last line, the term in the brackets is the "time rate of change of volume of an infinitesimally small element per volume - the defination for divergence of velocity, and hence can be written as

$$\int_{\mathcal{V}} \frac{D(\rho\, d\mathcal{V})}{Dt} \equiv \int_{\mathcal{V}} \frac{D\rho}{Dt}\, d\mathcal{V} + \int_{\mathcal{V}} (\rho \nabla \cdot \mathbf{V})\, d\mathcal{V} = 0 \tag{13}$$

From the defination of substantial derivative, the first term in RHS. of the equation above is

$$\int_{\mathcal{V}} \frac{D\rho}{Dt}\, d\mathcal{V} = \int_{\mathcal{V}} \left[\frac{\partial \rho}{\partial t} + \mathbf{V} \cdot \nabla \rho \right] d\mathcal{V} \tag{14}$$

Substituting the above expression into equation (13) we get

$$\int_{\mathcal{V}} \frac{D(\rho\, d\mathcal{V})}{Dt} \equiv \int_{\mathcal{V}} \left[\frac{\partial \rho}{\partial t} + \mathbf{V} \cdot \nabla \rho \right] d\mathcal{V} + \int_{\mathcal{V}} (\rho \nabla \cdot \mathbf{V})\, d\mathcal{V} = 0 \tag{15}$$

$$\equiv \int_{\mathcal{V}} \left[\frac{\partial \rho}{\partial t} + \underbrace{\mathbf{V} \cdot \nabla \rho + (\rho \nabla \cdot \mathbf{V})}_{\nabla \cdot \rho \mathbf{V}} \right] d\mathcal{V} = 0$$

$$\equiv \int_{\mathcal{V}} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{V} \right] d\mathcal{V} = 0 \tag{16}$$

Using the divergence theorem: $\int_{\mathcal{V}} (\nabla \cdot \rho \mathbf{V})\, d\mathcal{V} = \int_{\mathcal{S}} \rho \mathbf{V} \cdot d\mathbf{S}$, we can finally rewrite the last line in above equation as

$$\int_{\mathcal{V}} \frac{\partial \rho}{\partial t}\, d\mathcal{V} + \int_{\mathcal{S}} \rho \mathbf{V} \cdot d\mathbf{S} = 0 \equiv \text{LHS. in equation (9)} \tag{17}$$

We have thus successfully converted the various form of the same continuity equation via simple mathematical manipulations. This shows that the continuity equation derived from an identical mathematical model must be same.■