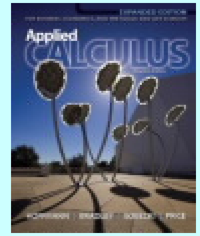
Chabot Mathematics

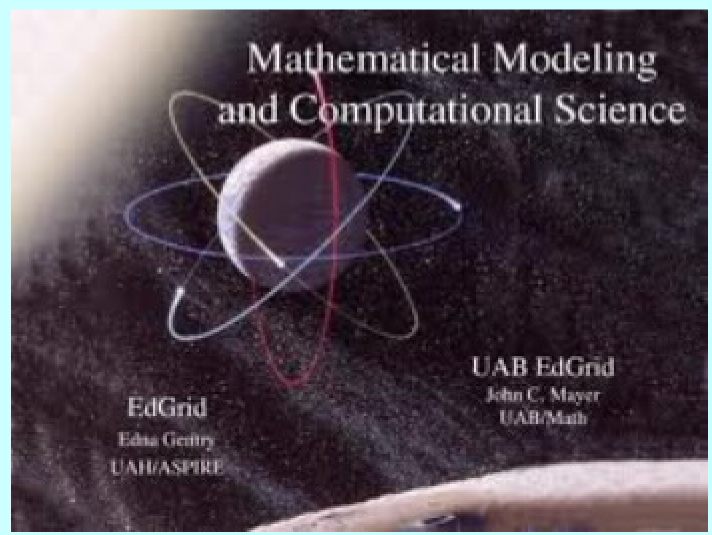
§1.5 Limits

Bruce Mayer, PE
Licensed Electrical & Mechanical Engineer
BMayer@ChabotCollege.edu

Review § 1.4



- Any QUESTIONS About
  - §1.4 → Functional Models
- Any QUESTIONS About HomeWork
  - §1.4 → HW-04



§1.5 Learning Goals

- Examine the limit concept and general properties of limits
- Compute limits using a variety of techniques
- Investigate limits involving infinity

∞

Limits

- Limits are a very basic aspect of **calculus** which needs to be taught first, after reviewing old material.
- The concept of limits is very important, since we will need to use limits to make new ideas and formulas in calculus.
- In order to understand calculus, limits are very fundamental to know!

Limits – Numerical Approx.

- A **limit** is a mere **trend**. This trend shows what number *y* is approaching when *x* is approaching some number!
- We can develop an intuitive notion of the **limit** of a function *f*(*x*) as *x* approaches some value *a*. Written as:

$$\lim_{x \to a} f(x) = ??$$

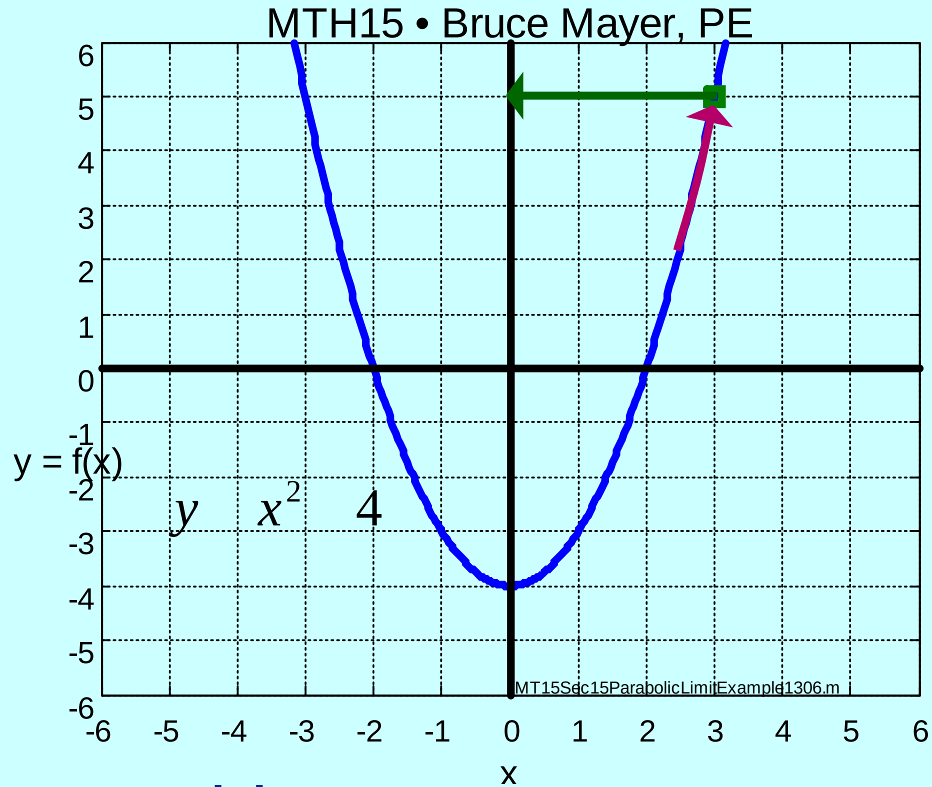
- Evaluate *f*(*x*) at values of *x* **close** to *a* and make an **educated guess** about the **trend**.

Limit Example

- Consider the Parabola at Right
- Notice that as *x* nears 3 that *y* approaches 5
- Write this behavior as

$$\lim_{x \to a} f(x) = ??$$

$$\lim_{x \to 3} x^2 - 4 = y$$



- In this case can evaluate limit by direct substitution

$$\lim_{x \to 3} (3)^2 - 4 = 5$$

Estimate using Limit Table

- The total cost, in $k, to produce *x* gallons of heavy-water (D_2O) can be estimated by the function:

$$C(x) \quad 3x \quad 130$$

- As the production level approaches 10 gallons, to what value does the **average** cost approach?

Estimate using Limit Table

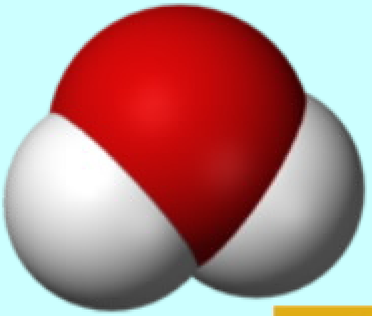
- **SOLUTION**
- The Average Cost is simply the Total Cost, $C(x)$, Divided by the Total Quantity, x, or: $AC(x) \quad \frac{C(x)}{x} \quad \frac{3x \quad 130}{x} \quad 3 \quad \frac{130}{x}.$
- Making a table of values near $x = 10$ we find

x

x	9	9.9	9.99	10.01	10.1	11
AC(x)	17.444	16.131	16.013	15.987	15.871	14.818

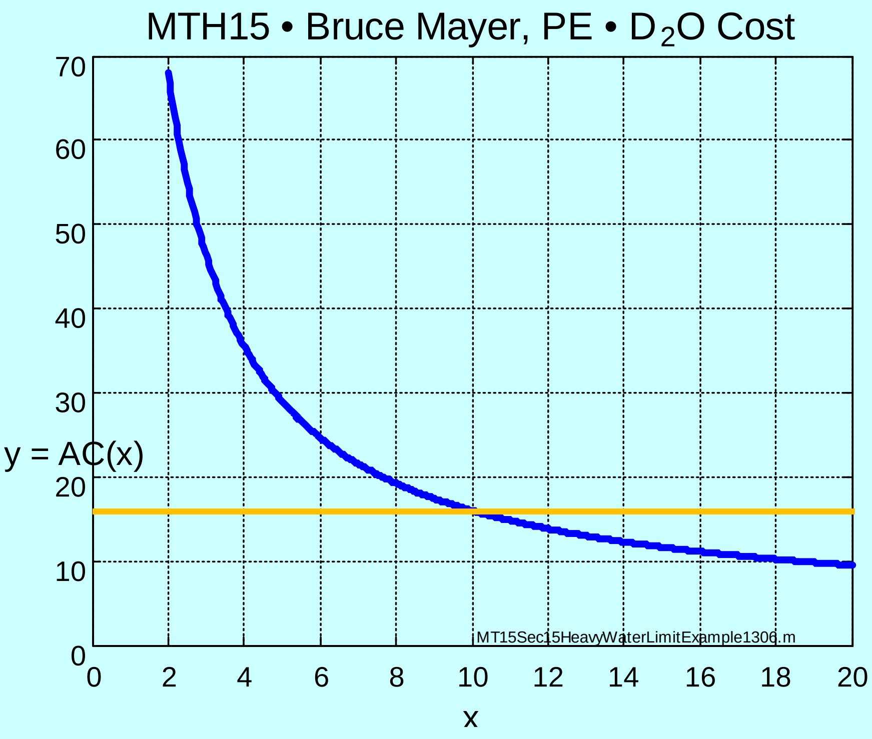
Estimate using Limit Table

- **SOLUTION**
- From these calculations, we conclude that as the desired volume of heavy water approaches 10 gallons, the average cost to produce it is approaching $16k dollars per gallon.
- Formally: $$\lim_{x \quad 10} AC(x) \quad 16$$



Estimate using Limit Table

- SOLUTION
- A graph of the average cost function can help us visualize the limit:
- Note that as we approach x=10, the graph seems to approach a height of 16.



Limits: SemiFormal Definition

- For a function *f*(*x*) if as *x* approaches some limiting value a, *f*(*x*) approaches some value *L*, write:

$$\lim_{x \quad a} f(x) \qquad L$$

- Which is read as:

“the limit of *f*(*x*) as *x* approaches *a* equals *L*”

Limit Properties (1)

- First assume that these limits exist:

$$\lim_{x \quad c} f \quad x \qquad \text{SomeNumber} \qquad \& \qquad \lim_{x \quad c} g \quad x \qquad \text{SomeNumber}$$

1. The Limit of a Sum equals the Sum of the Limits:

$$\lim_{x \quad c} f \quad x \qquad g \quad x \qquad\qquad \lim_{x \quad c} f \quad x \qquad \lim_{x \quad c} g \quad x$$

2. The Limit of a Difference equal the Difference of the Limits:

$$\lim_{x \quad c} f \quad x \qquad g \quad x \qquad\qquad \lim_{x \quad c} f \quad x \qquad \lim_{x \quad c} g \quad x$$

Limit Properties (2)

3. The Limit of a Product of a Constant & Fcn equals the Product of the Constant and Limit of the Fcn:

$$\lim_{x \quad c} k \quad f \quad x \qquad\qquad k \quad \lim_{x \quad c} f \quad x$$ For any Constant *k*

4. The Limit of a Product of Fcns equals the Product of the individual Limits:

$$\lim_{x \quad c} f \quad x \quad g \quad x \qquad\qquad \lim_{x \quad c} f \quad x \quad \lim_{x \quad c} g \quad x$$

Limit Properties (3)

5. The Limit of a Quotient of Fcns equals the Quotient of the individual Limits:

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} \quad \text{if} \quad \lim_{x \to c} g(x) \neq 0$$

6. The Limit of a Fcn to a Power equals the Limit raised to the Same Power

$$\lim_{x \to c} \left[f(x)\right]^{p} = \left[\lim_{x \to c} f(x)\right]^{p}$$

“Constant” Limits

- Some rather Obvious Limits
  - Limit of a Constant $$\lim_{x \quad c} k \qquad k$$
    - – *x* has NO EFFECT on *k*
  - Limit of approach to a Constant $$\lim_{x \quad c} x \qquad c$$
    - – As *x* approaches *c*, the Limit tends to *c* (Duh!)

Example ➡ Using Limit Props

- Evaluate the limit using Limit properties $z \quad \lim_{t \quad 1} t^4 \quad t \quad 1$
- **Solution**
- Use “Limit of Sum” Property

$$z \quad \lim_{t \quad 1} t^4 \quad \lim_{t \quad 1} t \quad \lim_{t \quad 1} 1$$

- Evaluate Last two Limits by Const Props

$$z \quad \lim_{t \quad 1} t^4 \qquad 1 \quad 1 \quad \lim_{t \quad 1} t^4 1 \;\; 1 \quad \lim_{t \quad 1} t^4 \quad 2$$

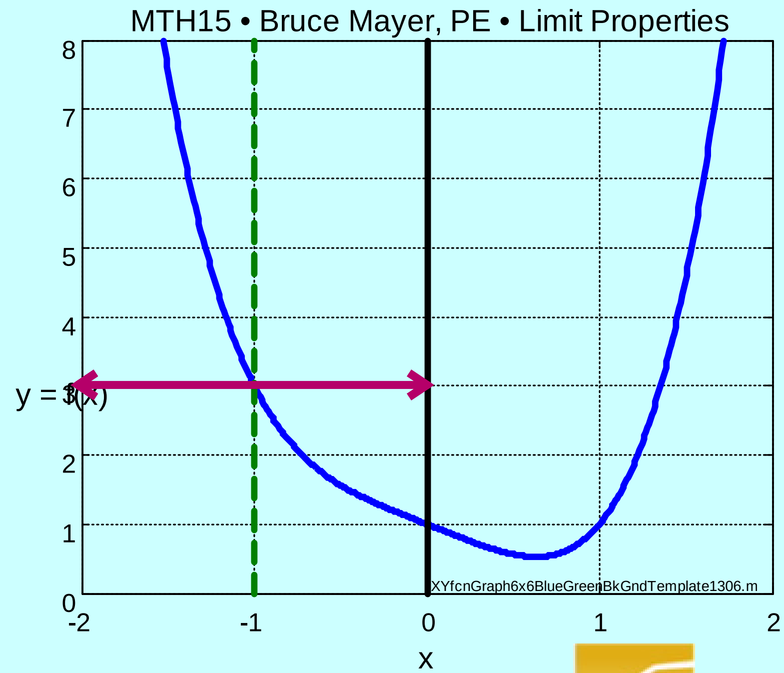
Example ➡ Using Limit Props

- Now use the “Limit to a Power” Property

$$z \quad \lim_{t \quad 1} t^{4} \quad 2 \quad \lim_{t \quad 1} t^{\ 4} \quad 2 \quad 1^{\ 4} \quad 2 \quad 1 \quad 2 \quad 3$$

- Thus Answer

$$z \quad \lim_{t \quad 1} t^{4} \quad t \quad 1 \quad 3$$



```
% Bruce Mayer, PE
% MTH-15 • 23Jun13
% XYfcnGraph6x6BlueGreenBkGndTemplate1306.m
% ref:
%
% The Limiest
xmin = -2; xmax = 2;   ymin = 0; ymax = 8;
% The FUNCTION
x = linspace(xmin,xmax,500); y = x.^4 - x + 1;
%
% The ZERO Lines
zxh = [xmin xmax]; zyh = [0 0]; zxv = [0 0]; zyv = [ymin ymax];
%
% the 6x6 Plot
axes; set(gca,'FontSize',12);
whitebg([0.8 1 1]); % Chg Plot BackGround to Blue-Green
plot(x,y, zxv,zyv, 'k', [-1,-1],[0,10],'--', 'LineWidth',
3),axis([xmin xmax ymin ymax]),...
    grid, xlabel('\fontsize{14}x'), ylabel('\fontsize{14}y =
f(x)'),...
    title(['\fontsize{14}MTH15 • Bruce Mayer, PE • Limit
Properties',]),...
    annotation('textbox',[.51 .05 .0 .1], 'FitBoxToText', 'on',
'EdgeColor', 'none', 'String',
'XYfcnGraph6x6BlueGreenBkGndTemplate1306.m','FontSize',7)
hold on
set(gca,'XTick',[-3:1:3]); set(gca,'YTick',[0:1:10])
```

MATLAB Code

Recall Rational Functions

- A **rational function** is a function $R(x)$ that is a **quotient** of two polynomials; that is,

$$R(x) = \frac{p(x)}{q(x)}$$

- Where
  - where $p(x)$ and $q(x)$ are polynomials and where $q(x)$ is **not** the zero polynomial.
  - The domain of R consists of all inputs x for which $q(x) \neq 0$.

Rational FUNCTION

- RATIONAL FUNCTION ≡ A function expressed in terms of rational expressions

- Example ➡ Find *f*(3) for this Rational Function:

$$f(x), \quad \frac{x^2 \;\; 3x \;\; 7}{x^2 \;\; 4}$$

- SOLUTION

$$f(x) \quad \frac{x^2 \;\; 3x \;\; 7}{x^2 \;\; 4}$$

$$f(3) \quad \frac{3^2 \;\; 3(3) \;\; 7}{(3)^2 \;\; 4}$$

$$\frac{9 \;\; 9 \;\; 7}{9 \;\; 5}$$

$$\frac{11}{4}$$

Limits of Rational Functions

- Given any rational function *R*(*x*) we claim that →

$$\lim_{x \quad a} R(x) \qquad R(a)$$

  - As long as *R*(*a*) is Defined

- Note that polynomials, *P*(*x*) are a type of rational function, so this theorem applies to them as well;
  - i.e: *P*(*x*) can be Written

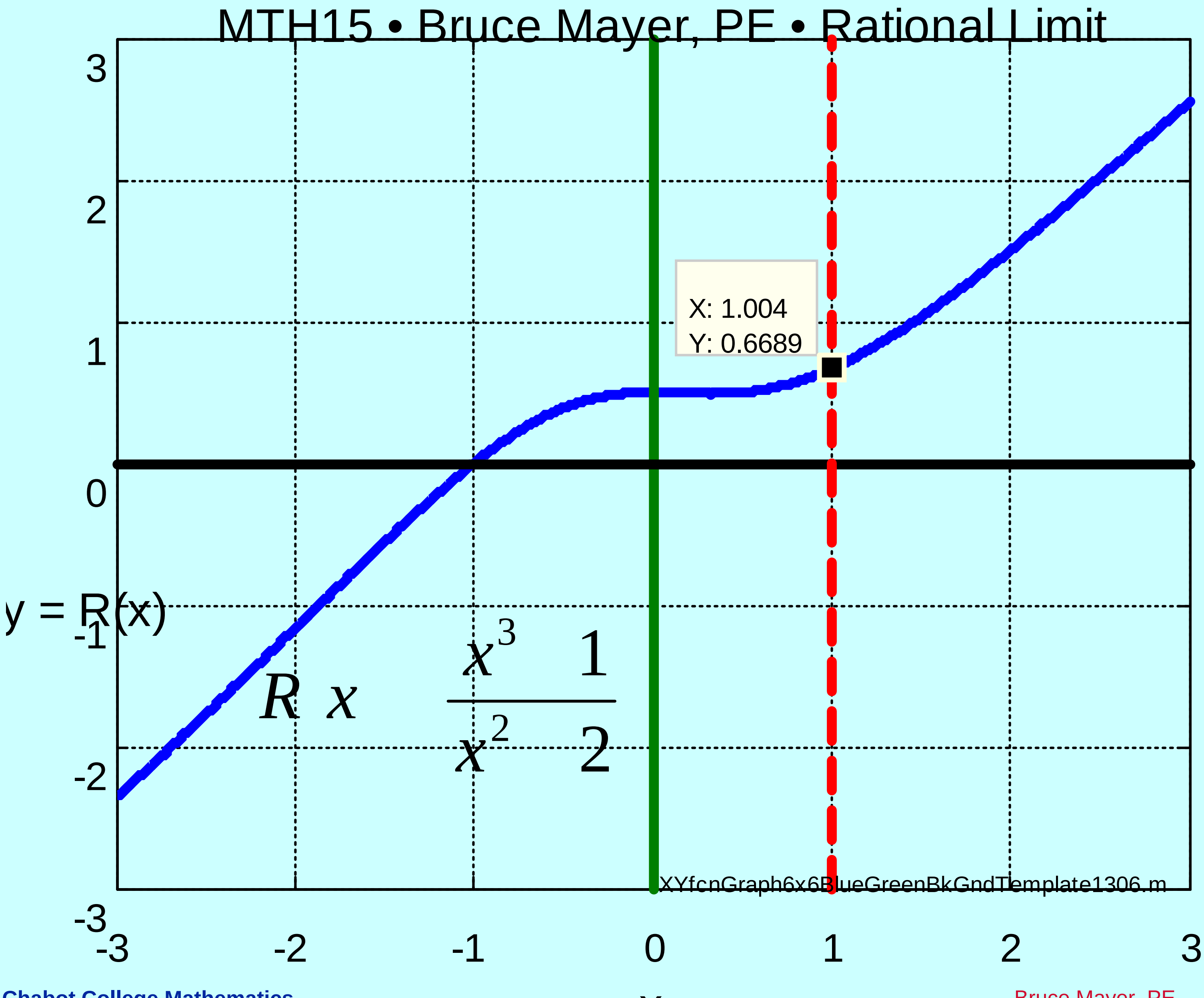
$$P(x) \qquad \frac{p(x)}{q \;\, x \qquad 1} \qquad \frac{p(x)}{1} \qquad p(x)$$

Example ➡ Rational Limit

- Find: $u \quad \lim_{x \quad 1} \frac{x^3 \quad 1}{x^2 \quad 2}$

- First note that the expression is defined at x = 1, Thus can use the Rational Limit theorem:

$$u \quad \lim_{x \quad 1} \frac{x^3 \quad 1}{x^2 \quad 2} \quad \frac{(1)^3 \quad 1}{(1)^2 \quad 2} \quad \frac{2}{3} \quad u$$



Rational Limit

MATLAB Code

```
% Bruce Mayer, PE
% MTH-15 • 23Jun13
% XYfcnGraph6x6BlueGreenBkGndTemplate1306.m
% ref:
%
% The Limits
xmin = -3; xmax = 3;   ymin = -3; ymax = 3;
% The FUNCTION
x = linspace(xmin,xmax,500); y1 = x.^3 + 1; y2 = x.^2 + 2; R =
y1./y2
%
% The ZERO Lines
zxh = [xmin xmax]; zyh = [0 0]; zxv = [0 0]; zyv = [ymin ymax];
%
% the 6x6 Plot
axes; set(gca,'FontSize',12);
whitebg([0.8 1 1]); % Chg Plot BackGround to Blue-Green
plot(x,R, zxv,zyv, zxh, zyh, 'k', [1,1],[-3,3],'--', 'LineWidth',
3),axis([xmin xmax ymin ymax]),...
    grid, xlabel('\fontsize{14}x'), ylabel('\fontsize{14}y =
R(x)'),...
    title(['\fontsize{14}MTH15 • Bruce Mayer, PE • Rational
Limit',]),...
    annotation('textbox',[.51 .05 .0 .1], 'FitBoxToText', 'on',
'EdgeColor', 'none', 'String',
'XYfcnGraph6x6BlueGreenBkGndTemplate1306.m','FontSize',7)
hold on
set(gca,'XTick',[-3:1:3]); set(gca,'YTick',[-3:1:3])
```

Example ➡ Limit by Algebra

- Two competing companies (the Chabot Co. & Gladiator, Inc.) have profit functions given as

Company	Profit Function
Chabot Co.	$P_{CC}(x) \quad x^2 \quad 4$
Gladiator Inc.	$P_{GI}(x) \quad x^2 \quad x \quad 6,$

  - Where x ≡kUnits sold

Example ➡ Limit by Algebra

- At What value does the **ratio** of profit for Chabot-Co to Gladiator-Inc approach as sales approaches 2000 units?
- SOLUTION
- Need to Find: $R_P \quad \lim_{x \quad 2} \frac{P_{CC}(x)}{P_{GI}(x)} \quad \lim_{x \quad 2} \frac{x^2 \quad 4}{x^2 \quad x \quad 6}$
- First, note that the rational functions Limit Theorem **does not apply** as the Limit Approaches 0/0 →

$$\lim_{x \quad 2} \frac{x^2 \quad 4}{x^2 \quad x \quad 6} \qquad \frac{2^2 \quad 4}{2^2 \quad 2 \quad 6} \quad \frac{0}{0}!!!$$

Example ➨ Limit by Algebra

- Having the limiting value be of the **indeterminant** form 0/0 often reveals that **algebraic simplification** would be of assistance
- Notice that (x − 2) is a factor of the Dividend & Divisor:

$$R_p = \lim_{x \to 2} \frac{P_{CC}(x)}{P_{CI}(x)} = \lim_{x \to 2} \frac{x^2 - 4}{x^2 + x - 6} = \lim_{x \to 2} \frac{(x+2)(x-2)}{(x-2)(x+3)}.$$

Example ➡ Limit by Algebra

- Factoring and Symplifying $R_P \quad \lim_{x \quad 2} \frac{\cancel{(x \quad 2)}(x \quad 2)}{\cancel{(x \quad 2)}(x \quad 3)} \quad \lim_{x \quad 2} \frac{x \quad 2}{x \quad 3},$
- This Produces a limit for which the theorem about rationals DOES apply:

$$R_P \quad \lim_{x \quad 2} \frac{x \quad 2}{x \quad 3} \quad \frac{2 \quad 2}{2 \quad 3} \quad 0.80$$

- In other words, the profit for **Chabot-Co.** approaches 80% that of **Gladiator-Inc.** as sales approach 2000 units.

Limits as *x* grows W/O Bound

- As *x* **increases** WithOut bound, if *f*(*x*) approaches some value *L* then write:

$$\lim_{x} \; f \;\; x \qquad L$$

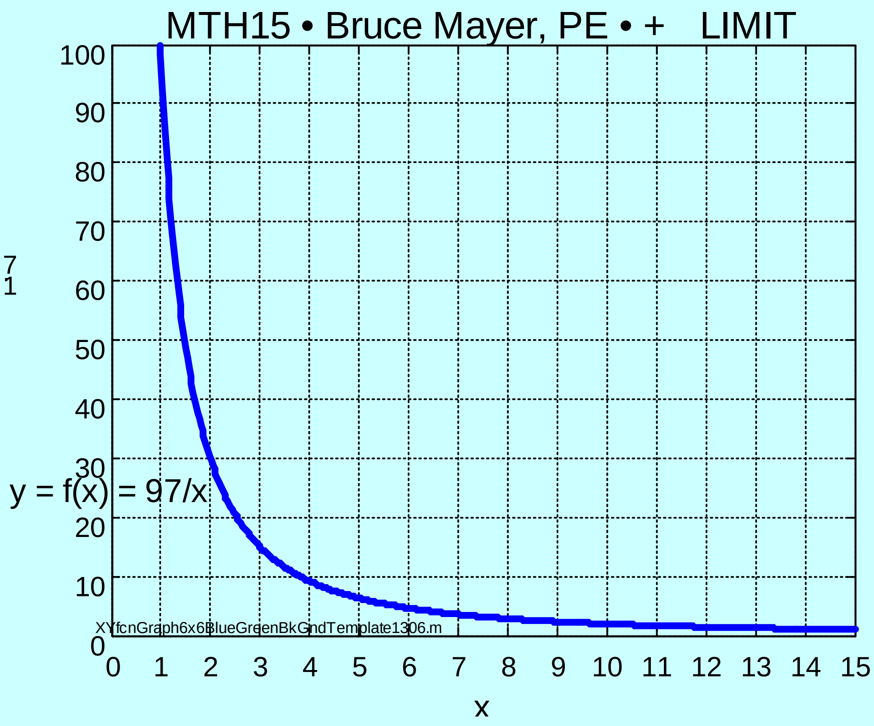
- Similarly, if as *x* **decreases** WithOut bound, *f*(*x*) approaches some value *K* then write:

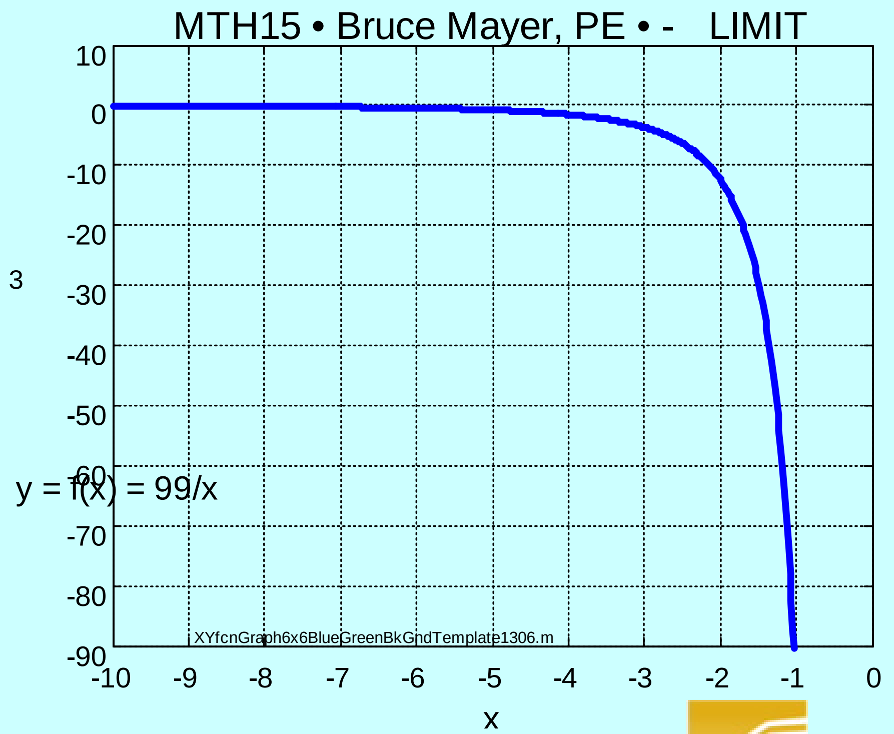
$$\lim_{x} \; f \;\; x \qquad K$$

- Note that the rules for limits introduced earlier apply; e.g., All Limits EXIST

∞Limits Graphically

- As x increases Toward $+\infty$, $f(x) \rightarrow +0$
- As x increases Toward $-\infty$, $f(x) \rightarrow -0$





+ MATLAB Code

```
% Bruce Mayer, PE
% MTH-15 • 23Jun13
% XYfcnGraph6x6BlueGreenBkGndTemplate1306.m
% ref:
%
% The Limits
xmin = 0; xmax =15;   ymin = 0; ymax = 100;
% The FUNCTION
x = linspace(xmin,xmax,500); y = 97./x.^1.7;
%
% The ZERO Lines
zxh = [xmin xmax]; zyh = [0 0]; zxv = [0 0]; zyv = [ymin ymax];
%
% the 6x6 Plot
axes; set(gca,'FontSize',12);
whitebg([0.8 1 1]); % Chg Plot BackGround to Blue-Green
plot(x,y, 'LineWidth', 3),axis([xmin xmax ymin ymax]),...
    grid, xlabel('\fontsize{14}x'), ylabel('\fontsize{14}y = f(x)
= 97/x^1^.^7'),...
    title(['\fontsize{16}MTH15 • Bruce Mayer, PE • +\infty
LIMIT',]),...
    annotation('textbox',[.1 .05 .0 .1], 'FitBoxToText', 'on',
'EdgeColor', 'none', 'String',
'XYfcnGraph6x6BlueGreenBkGndTemplate1306.m','FontSize',7)
hold on
set(gca,'XTick',[xmin:1:xmax]); set(gca,'YTick',[ymin:10:ymax])
```

– MATLAB Code

```
% Bruce Mayer, PE
% MTH-15 • 23Jun13
% XYfcnGraph6x6BlueGreenBkGndTemplate1306.m
%
clear; clc;
%
% The Limits
xmin = -10; xmax = 0;   ymin = -100; ymax = 0;
% The FUNCTION
x = linspace(xmin,xmax,500); y = 99./x.^3;
%
% The ZERO Lines
zxh = [xmin xmax]; zyh = [0 0]; zxv = [0 0]; zyv = [ymin ymax];
%
% the 6x6 Plot
axes; set(gca,'FontSize',12);
whitebg([0.8 1 1]); % Chg Plot BackGround to Blue-Green
plot(x,y, 'LineWidth', 3),axis([xmin xmax ymin+10 ymax+10]),...
    grid, xlabel('\fontsize{14}x'), ylabel('\fontsize{14}y = f(x)
= 99/x^3'),...
    title(['\fontsize{16}MTH15 • Bruce Mayer, PE • -\infty
LIMIT',]),...
    annotation('textbox',[.2 .05 .0 .1], 'FitBoxToText', 'on',
'EdgeColor', 'none', 'String',
'XYfcnGraph6x6BlueGreenBkGndTemplate1306.m','FontSize',7)
hold on
set(gca,'XTick',[xmin:1:xmax]); set(gca,'YTick',[ymin:10:ymax+10])
```

Example ➡ Evaluate an Infinity Limit

- The total cost, in \$k, to produce *x* gallons of heavy water (D_2O) can be estimated by the function:

$$C(x) \quad 3x \quad 130$$

- As the production level grows WITHOUT BOUND, to what value(s) does the **average** cost approach?

Example ➡ Evaluate an Infinity Limit

- **SOLUTION**
- The Limit to Evaluate: $AC \qquad \lim_{x} \frac{C \;\; x}{x} \quad \lim_{x} \frac{3x \quad 130}{x}$
- We can either simplify by algebra or use a general strategy for limits at infinity for rational functions; try This:
  - Divide numerator and denominator by the term in the denominator having the largest exponent.

Example ➡ Evaluate an Infinity Limit

- Divide Top & Bot by *x*

$$AC \qquad \lim_{x} \frac{3x \quad 130}{x} \qquad \lim_{x} \frac{(3x \quad 130)/x}{x/x} \qquad \lim_{x} \frac{3 \quad 130/x}{1}$$

- Using Sum-of-Limits and Const-times-Limit find:

$$AC \qquad \lim_{x} \; 3 \quad \frac{130}{x} \qquad 3 \quad 130 \quad \lim_{x} \frac{1}{x} \qquad 3 \quad 130(0) \qquad 3.$$

- State: As the volume of Heavy Water increases without bound, average cost approaches $3k per gallon.

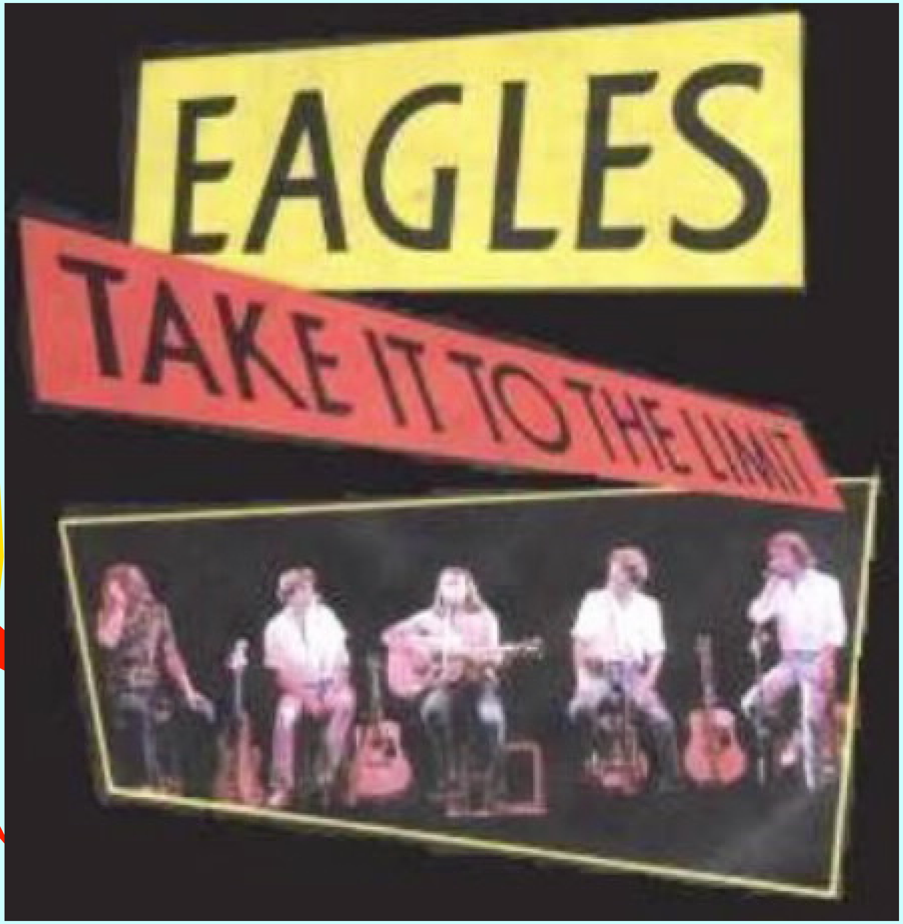
WhiteBoard Work

- Problem §1.5-56: New Employee Productivity
- *n* items produced after *t* weeks on the job:
  - Employees are paid 20¢ per item

$$n \quad 70 \quad \frac{150}{t \quad 4}$$

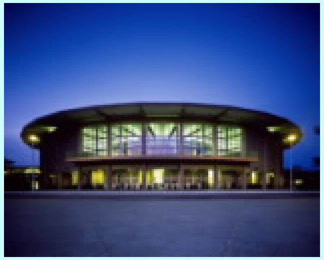
All Done for Today

So put me on
a highway
and show me a sign
and TAKE IT
TO THE LIMIT
one more time



Appendix

$$r^2 \quad s^2 \qquad r \quad s \quad r \quad s$$



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Limit Property ShortHand

- Limit of a Sum (or Difference) equals the Sum (or Difference) of Limits

$$\lim_{x\to c}\left[f(x)+g(x)\right]=\lim_{x\to c}f(x)+\lim_{x\to c}g(x)$$

$$\lim_{x\to c}\left[f(x)-g(x)\right]=\lim_{x\to c}f(x)-\lim_{x\to c}g(x)$$

- The Limit of a Const times a Fcn equal the Const times the Limit of the Fcn

$$\lim_{x\to c}\left[k\cdot f(x)\right]=k\cdot\lim_{x\to c}f(x)$$

Limit Property ShortHand

- Limit of a Product (or Quotient) equals the Product (or Quotient) of the Limits

$$\lim_{x \to c} \left[f(x) \cdot g(x) \right] = \left[\lim_{x \to c} f(x) \right] \cdot \left[\lim_{x \to c} g(x) \right]$$

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} \quad \text{if } \lim_{x \to c} g(x) \neq 0$$

- The Limit of a Power equals the Power of the Limit

$$\lim_{x \to c} \left[f(x) \right]^p = \left[\lim_{x \to c} f(x) \right]^p$$

1.5-56

$$n(t) = 70 - 150/(t+4)$$

WHERE

$n \equiv$ NO. ITEMS PRODUCED

$t \equiv$ WEEKS ON THE JOB

AMOUNT, $A(n)$, PAID TO EMPLOYEE IS 20¢/ITEM

a) FIND $A(t)$

$$A(n) = \frac{\$0.2}{\text{ITEM}} \times n$$

SUBBING FROM ABOVE

$$A(t) = 0.2[n] = 0.2\left[70 - \frac{150}{t+4}\right]$$

ANS (a)

$$\boxed{A(t) = \$14 - \$\frac{150 \text{ WK}}{t + 4\text{WK}}}$$

LONG RUN EARNING; i.e.,

$t \to \infty$

$$\lim_{t \to \infty} 70 - \frac{150}{t+4} = A_{MAX}$$

BY LIMIT PROPERTIES

$$A_{MAX} = \lim 70 - \lim \frac{150}{t+4}$$

$$= 70 - \lim_{t \to \infty} \frac{150}{t+4}$$

$$= \$70 - \lim_{t \to \infty} \frac{150}{\infty + 4}$$

$$= \$70 - 0$$

ANS (b)

$$A_{MAX} = \$70$$