<u>Binomial Expansion</u>

<u>Refreshment on Binomial Theorem from "A-Math"</u>

From A-Math in Secondary School, we learnt that whenever the power of n is a **positive whole number** (i.e. $n \in \mathbb{N}$), the expansion of **2 terms** can be expanded according to the formula:

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \binom{n}{3} a^{n-3} b^3 + \cdots + b^n$$

Can be written as: $\binom{n}{0} a^{n-0} b^0$

Can be written as: $\binom{n}{n} a^{n-n} b^n$

Where $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ and $r! = r(r-1)(r-2)(r-3)\ldots(3)(2)(1)$

This is known as: "n choose r". Which is the number of ways to choose r objects out from n distinct objects. (We will learn these more precisely in J2)

$$(a+b)^n = \underbrace{a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \binom{n}{3} a^{n-3} b^3 + \cdots + b^n}_{\text{There are } (n+1) \text{ terms}}$$

"n" is a **<u>positive whole number</u>**

If we wish to find what a term is for a particular power of x without expanding the whole binomial expansion, we could use what we call the T_{r+1} (a.k.a the $(r+1)^{th}$ term):

$$T_{r+1} = \binom{n}{r} a^{n-r} b^r$$

Constants

x-term

In the expansion of $(a + bx)^n$, the expansion will be in ascending powers of x.

In the event where 'a' and 'b' in $(a+b)^n$ contains both x-terms and we want to find the coefficient of a x-term with a particular power, we could use the T_{r+1} too.

<u>Example 1:</u> (This is a common question in A-math)

Find the term <u>independent of x</u> in $\left(x^2 - \frac{1}{x^2}\right)^{16}$.

This means the term with **no** x (i.e. the x^0 term)

<u>Solution:</u>

We no need to expand the entire expansion for $\left(x^2 - \frac{1}{x^2}\right)^{16}$. By making use of the T_{r+1}, we can formulate what the $(r+1)^{th}$ term in the expansion will be in terms of r.

$T_{r+1} = \binom{n}{r} a^{n-r} b^r$. Here in $\left(x^2 - \frac{1}{x^2}\right)^{16}$, $a = x^2$ and $b = -\frac{1}{x^2}$.

Substituting these into T_{r+1} gives:

$$\begin{aligned} T_{r+1} &= \binom{16}{r} (x^2)^{16-r} \left(-\frac{1}{x^2}\right)^r \\ &= \binom{16}{r} x^{32-2r} (-1)^r x^{-2r} \\ &= \binom{16}{r} (-1)^r x^{32-4r} \\ &= \binom{16}{r} (-1)^r x^{32-4r} \end{aligned}$$

Expand out using the law of indices

Simplify the x-terms using law of indices

For the term independent of x, the power of x is "0". So we set the power in T_{r+1} to "0" and solve for r:

$$32 - 4r = 0 \Rightarrow r = 8$$

This means that when $r = 8$ (i.e. the 9th term in the expansion will be the term independent of x):

$$\therefore T_{r+1} = \binom{16}{r} (-1)^8 = \underbrace{12870}_{\text{This is the term independent of } x}$$

This formula is inside the "MF-15" Formula List

Binomial Series Expansion Formula:

When the power "n" in $(a+b)^n$ is **not** a positive integer, we have to expand using a different formula:

n must **not** be a positive integer

$$(1+x)^n = \underbrace{1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots}_{} \quad \text{for } |x| < 1$$

Must be "1" to apply this formula directly

There are infinite number of terms in **ascending powers of x**

Note that:

- n is NOT a positive integer (i.e. n could be a positive/negative fraction or a negative whole number)

- There is an infinite number of terms in the expansion instead of $(n+1)$ terms as compared with the binomial theorem from A-Math. But in a question, we will only be ask to expand till a certain degree of x only. (Of course!)

- Since we are adding up an infinite numbers of terms involving x, we need to state what range of values x could be so that the infinite terms adds up to a finite number (Here, we say that the infinite sum "converges")

Known also as the range of validity of the binomial expansion

In the event where the constant term is **not** "1", we have to factor it out to make it "1" as follows:

Expand using the binomial expansion formula

$$(a + bx)^n = \left[a\left(1 + \frac{bx}{a}\right)\right]^n = a^n \boxed{\left(1 + \frac{bx}{a}\right)^n}$$

for $\left|\frac{bx}{a}\right| < 1$

a is any number that is **not** "1"

Expand the power in using the law of indices

Example 2:

Expand $(4 - 3x)^{-2}$ up to and including the x^3-term. State the range of validity for the expansion.

Solution: Factor out a "4" to make it a "1"

Apply Binomial Expansion

$$(4 - 3x)^{-2} = \left[4\left(1 - \frac{3x}{4}\right)\right]^{-2} = 4^{-2}\boxed{\left(1 - \frac{3x}{4}\right)^{-2}}$$

$$= \frac{1}{16}\left[1 + (-2)\left(-\frac{3x}{4}\right) + \frac{(-2)(-3)}{2!}\left(-\frac{3x}{4}\right)^2 + \frac{(-2)(-3)(-4)}{3!}\left(-\frac{3x}{4}\right)^3 + \cdots\right]$$

$$= \frac{1}{16}\left[1 + (2)\left(\frac{3x}{2}\right) + 3\left(\frac{9x^2}{16}\right) + (-4)\left(-\frac{27x^3}{64}\right) + \cdots\right]$$

$$= \frac{1}{16}\left[1 + 3x + \frac{27x^2}{16} + \frac{27}{16}x^3 + \cdots\right]$$

$$= \frac{1}{16} + \frac{3}{16}x + \frac{27}{256}x^2 + \frac{27}{256}x^3 + \cdots$$

The range of validity is: $\left|\frac{3x}{4}\right| < 1$

$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

$$\Rightarrow \frac{|3x|}{|4|} < 1$$

$\times 4$

$$\Rightarrow |3x| < 4$$

$$\Rightarrow |x| < \frac{4}{3} \Rightarrow -\frac{4}{3} < x < \frac{4}{3}$$

Binomial Expansion can be used to approximate the value of certain numbers that otherwise cannot be expressed as a fraction (i.e. a rational term) to a fraction. In short, **binomial expansion provides us with a way to approximate an irrational number to a rational number** <u>by letting x be a particular *rational* value (the value of x is usually given in the question)</u>

There is 2 rules that we must know with regards to the value of x chosen:

- The value of x **must** be within the range of validity
- The **closer the value of x is to "0"** (within the range of validity that is), the **better** the approximation

This is illustrated in the next example.

★ **Example 3:**

This means that we will be expanding till x^2 term only

Given that x is sufficiently small for x^3 and higher power of x to be neglected, find the series expansion of $\frac{\sqrt{1+x}}{2x-1}$ in ascending powers of x, up to and including the term in x^2. State the range of values of x for which this expansion is valid.

To apply binomial expansion for $\frac{\sqrt{1+x}}{2x-1}$, we need to express $\frac{\sqrt{1+x}}{2x-1}$ in power form:

$$\frac{\sqrt{1+x}}{2x-1} = (1+x)^{\frac{1}{2}}(2x-1)^{-1}$$

By letting:

(i) $x = \frac{1}{9}$,

(ii) $x = \frac{13}{32}$,

In each case, obtain an approximate for the value of $\sqrt{10}$ as a fraction in simplest form. Without the use of a calculator, which will result in a better approximation? Briefly explain your choice.

State, with reason whether $x = 9$ can be used to approximate the value of $\sqrt{10}$.

Solution:

Since the power are both not positive integers, we have to apply binomial expansion to both. (We must ensure constant terms are both "1"!)

$$\frac{\sqrt{1+x}}{2x-1} = (1+x)^{\boxed{\frac{1}{2}}}(2x-1)^{\boxed{-1}}$$

$$= (1+x)^{\frac{1}{2}}[-1(1-2x)]^{-1}$$

$$= (1+x)^{\frac{1}{2}}(-1)^{-1}(1-2x)^{-1}$$

$$= -\left(1+\left(\frac{1}{2}\right)(x)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}(x)^2+\cdots\right)\left(1+(-1)(-2x)+\frac{(-1)(-2)}{2!}(-2x)^2+\cdots\right)$$

$$= -\left(1+\frac{x}{2}-\frac{1}{8}x^2+\cdots\right)(1+2x+4x^2+\cdots)$$

$$= -(1+2x+4x^2+\frac{x}{2}+x^2-\frac{1}{8}x^2+\cdots)$$

$$= -(1+\frac{5x}{2}+\frac{39}{8}x^2+\cdots)$$

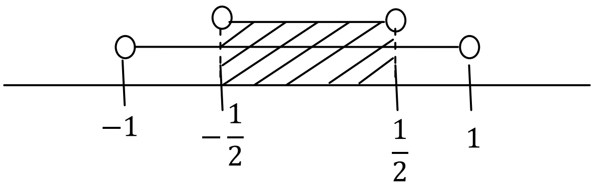
$$= -1-\frac{5x}{2}-\frac{39}{8}x^2+\cdots$$

Since we perform 2 binomial expansions (1 on $(1+x)^{\frac{1}{2}}$ and another on $(2x-1)^{-1}$), there will be 2 "range of validity" for each of the respective binomial expansion:

Range of validity for $(1+x)^{\frac{1}{2}}$: $|x| < 1 \implies -1 < x < 1$

Range of validity for $(2x-1)^{-1}$: $|2x| < 1 \implies |x| < \frac{1}{2} \implies -\frac{1}{2} < x < \frac{1}{2}$

But since the 2 separate binomial expansion is considered as a single binomial expansion for $\frac{\sqrt{1+x}}{2x-1}$, we need to combined the 2 "range of validity" into a single range:



$\therefore$ Range of validity for $\frac{\sqrt{1+x}}{2x-1}$: $-\frac{1}{2} < x < \frac{1}{2}$.

(i) Substitute $x = \frac{1}{9}$ into: $\frac{\sqrt{1+x}}{2x-1} = -1 - \frac{5x}{2} - \frac{39}{8}x^2 + \cdots$ gives:

$$\frac{\sqrt{1+\frac{1}{9}}}{2\left(\frac{1}{9}\right)-1} = -1 - \frac{5}{2}\left(\frac{1}{9}\right) - \frac{39}{8}\left(\frac{1}{9}\right)^2 + \cdots$$

$$\frac{\sqrt{\frac{10}{9}}}{\left(-\frac{7}{9}\right)} \approx -1 - \frac{5}{18} - \frac{13}{216}$$

$$\frac{3\sqrt{10}}{-7} \approx -\frac{289}{216}$$

$$\sqrt{10} \approx \frac{2023}{648}$$

(ii) Substitute $x = \frac{13}{32}$ into $\frac{\sqrt{1+x}}{2x-1} = -1 - \frac{5x}{2} - \frac{39}{8}x^2 + \cdots$ gives:

$$\frac{\sqrt{1+\frac{13}{32}}}{2\left(\frac{13}{32}\right)-1} = -1 - \left(\frac{5}{2}\right)\left(\frac{13}{32}\right) - \left(\frac{39}{8}\right)\left(\frac{13}{32}\right)^2 + \cdots$$

$$\frac{\sqrt{\frac{45}{32}}}{\left(-\frac{3}{16}\right)} \approx -1 - \frac{65}{64} - \frac{6591}{8192}$$

$$\frac{\sqrt{\frac{90}{64}}}{\left(-\frac{3}{16}\right)} \approx -\frac{23103}{8192}$$

$$-\frac{16\left(\frac{\sqrt{90}}{8}\right)}{3} \approx -\frac{23103}{8192}$$

$$-\frac{16\left(\frac{3\sqrt{10}}{8}\right)}{3} \approx -\frac{23103}{8192} \Longrightarrow 2\sqrt{10} = \frac{23103}{8192}$$

$$\therefore \sqrt{10} \approx \frac{38505}{16384}$$

Since $x = \frac{1}{9}$ is much closer to "0" than $x = \frac{13}{32}$, thus $x = \frac{1}{9}$ will give a better approximation than $x = \frac{13}{32}$. (Note that $x = \frac{1}{9}$ and $x = \frac{13}{32}$ are both valid approximations as they are both within the range of validity)

$x = 9$ **cannot** be used as an approximation to $\sqrt{10}$ using the binomial series expansion as it is **not** within the range of validity (which is $-\frac{1}{2} < x < \frac{1}{2}$)

<u>Binomial Series Expansion in Descending powers of x:</u>

In the earlier formula, the binomial series expansion:

$$(1+x)^n = \underbrace{1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots}_{\text{Ascending powers of } x}$$

also known as ascending powers of $\frac{1}{x}$

If we are going to apply binomial series expansion in **descending powers of x**, we have to make factor out a term from $(1+x)^n$. **The term to factor out the x-term from** $(1+x)^n$**:**

$$(1+x)^n = \left[x\left(1+\frac{1}{x}\right)\right]^n = x^n \boxed{\left(1+\frac{1}{x}\right)^n}$$

Perform binomial expansion on this

$$= x^n\left[1 + (n)\left(\frac{1}{x}\right) + \frac{n(n-1)}{2}\left(\frac{1}{x}\right)^2 + \cdots\right.$$

$$= x^n \underbrace{\left[1 + (n)\left(\frac{1}{x}\right) + \frac{n(n-1)}{2}\left(\frac{1}{x}\right)^2 + \cdots\right]}_{\text{Descending powers of } x} \quad \text{for } \left|\frac{1}{x}\right| < 1$$

In the general case of expanding $(a + bx)^n$ in descending powers of x, we will expand the x-term inside the bracket, which is bx:

$$(a + bx)^n = \left[bx\left(1 + \frac{a}{bx}\right)\right]^n = (bx)^n \boxed{\left(1 + \frac{a}{bx}\right)^n}$$

Perform binomial expansion on this!

The following example illustrates the application of binomial series expansion in descending powers of x.

Example 4:

which is "descending powers of x"

Expand $\frac{1}{\sqrt{2+3x}}$ in $\boxed{\text{ascending powers of } \frac{1}{x}}$ up to and including x^{-2}.

State the range of validity for the above expansion.

Solution:

Express $\frac{1}{\sqrt{2+3x}}$ in power form:

$$\frac{1}{\sqrt{2+3x}} = (2+3x)^{-\frac{1}{2}}$$

$$\begin{aligned}
\frac{1}{\sqrt{2+3x}} &= (2+3x)^{-\frac{1}{2}} \\
&= \left[3x\left(\frac{2}{3x} + 1\right)\right]^{-\frac{1}{2}} \\
&= (3x)^{\frac{1}{2}}\left(1 + \frac{2}{3x}\right)^{-\frac{1}{2}} \\
&= (3x)^{\frac{1}{2}}\left[1 + \left(-\frac{1}{2}\right)\left(\frac{2}{3x}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\frac{2}{3x}\right)^2 + \cdots\right] \\
&= (3x)^{\frac{1}{2}}\left[1 - \frac{1}{3x} + \left(\frac{3}{8}\right)\left(\frac{4}{9x^2}\right) + \cdots\right] \\
&= \left(\sqrt{3}x^{\frac{1}{2}}\right)\left[1 - \frac{1}{3x} + \frac{1}{6x^2} + \cdots\right] \\
&= \sqrt{3}x^{\frac{1}{2}} - \frac{1}{\sqrt{3}}x^{-\frac{1}{2}} + \frac{\sqrt{3}}{6}x^{-\frac{3}{2}} + \cdots
\end{aligned}$$

Realize x has to be positive since:

- There is the presence of $\sqrt{x}$ (as we cannot square root a negative number)
- x cannot be "0" as there is a division by $\sqrt{x}$

Range of validity: $\left|\frac{2}{3x}\right| < 1 \implies \frac{2}{|3x|} < 1 \implies 2 < |3x| \implies \frac{2}{3} < |x|$

$$\implies \boxed{x > \frac{2}{3}} \text{ or } x < -\frac{2}{3} \text{ (rej.)}$$

This is the only answer

Finding the general term of a binomial series expansion

Finding the general term refers to finding what exactly is the x^n-term without expanding all the terms till that term (i.e. if we are interested in finding the coefficient of the x^{20}-term, we do **not** have to perform the binomial series expansion from "1" to x^{20})

We can obtain the general term for a binomial series expansion by using the formula given in the "MF-15" Formula list:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + \boxed{\frac{n(n-1)(n-2)\dots(n-r+1)}{r!}x^r}$$

This is the general term

Here:

- n is the power for $(1+x)^n$
- r is the power of the general term

A few things about the general term:

There are n numbers being multiplied on the numerator **starting from $\boldsymbol{n}$**

$$\frac{\overbrace{n(n-1)(n-2)\dots(n-r+1)}}{r!}x^r$$

This is the same number (which the the general term)

If the question requests us to find the x^n-term (i.e. the general term), the variable "r" above is my "n" (as "n" is now the power of the general term).

The next example illustrates this.

Example 5:

Find the x^n term (i.e. the general term) in the series expansion of $(2-3x)^{-2}$ and hence find the coefficient of x^{500}, giving the answer in exact exponential form.

Solution:

To apply the general term formula, we must ensure the constant term in $(2-3x)^{-2}$ is "1". So:

$$(2-3x)^{-2} = \left[2\left(1-\frac{3}{2}x\right)\right]^{-2} = 2^{-2}\boxed{\left(1-\frac{3}{2}x\right)^{-2}}$$

We shall apply the formula for the general term for this one

General term for $\left(1-\frac{3}{2}x\right)^{-2}$: $\dfrac{(-2)(-3)(-4)\dots(-2-n+1)}{n!}\left(-\dfrac{3}{2}x\right)^n$

We shall simplify the above "horrendous" looking term.

$$\frac{\overbrace{(-2)(-3)(-4)\ldots(-2-n+1)}^{\text{Product of } n \text{ numbers}}}{n!}\boxed{\left(-\frac{3}{2}x\right)^n} \qquad \left(-\frac{3}{2}\right)^n = (-1)^n\left(\frac{3}{2}\right)^n$$

$$= \frac{(-2)(-3)(-4)\ldots(-n-1)}{n!}(-1)^n\left(\frac{3}{2}\right)^n x^n$$

$$= \frac{[\overbrace{(-1)(2)}^{-2}][\overbrace{(-1)(3)}^{-3}][\overbrace{(-1)(4)}^{-4}]\ldots[\overbrace{(-1)(n+1)}^{-n-1}]}{n!}(-1)^n\left(\frac{3}{2}\right)^n x^n$$

$$= \frac{\boxed{(-1)^n}(2)(3)(4)\ldots(n+1)}{n!}(-1)^n\left(\frac{3}{2}\right)^n x^n$$

This is due to the product of n number of "-1"

$$= \frac{(-1)^n\cancel{(2)}\cancel{(3)}\cancel{(4)}\ldots\cancel{(n)}(n+1)}{(1)\cancel{(2)}\cancel{(3)}\cancel{(4)}\ldots\cancel{(n)}}(-1)^n\left(\frac{3}{2}\right)^n x^n$$

$$= (-1)^n(n+1)(-1)^n\left(\frac{3}{2}\right)^n x^n$$

$(-1)^n(-1)^n = (-1)^{2n} = [(-1)^2]^n = 1^n = 1$

$$= (n+1)\left(\frac{3}{2}\right)^n x^n$$ (This is the answer)

To find the coefficient of x^{500}, we let $n = 500$ in the above answer:

$\therefore$ Coefficient of $x^{500} = (500+1)\left(\frac{3}{2}\right)^{500} = (501)\left(\frac{3}{2}\right)^{500}$