Solution to Practice Problems
Math 3311

Section 0:

Exercise 4: Describe the set A by listing its elements, where

$$A = \left\{ m \in \mathbb{Z} \mid m^2 - 4m < 15 \right\}.$$

Solution: The set A is defined using the characterizing property

$$P(m): \text{"}m^2 - 4m < 15\text{"}, \text{ where } m \in \mathbb{Z}.$$

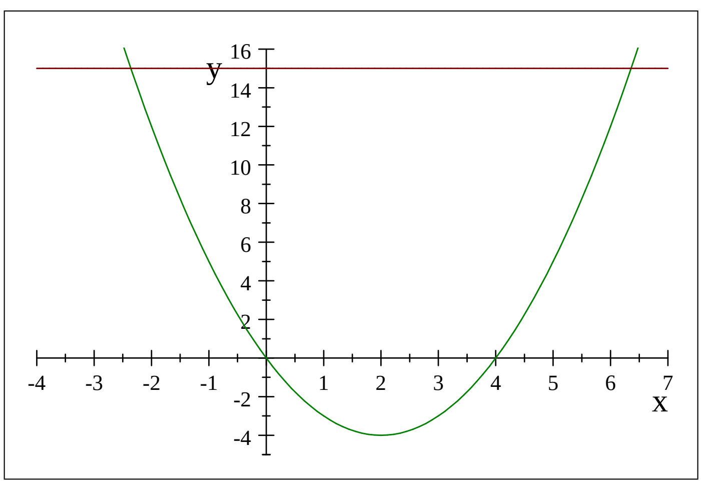
In order to find its elements, we need to find all integers $m \in \mathbb{Z}$ for which $P(m)$ is true. Since, $m \in \mathbb{Z}$ we can start by checking successively integers indicating in the last column whether $m^2 - 4m < 15$ holds for m by writing T (true) if the statement is true and F (false) when the statement is false:

m	$m^2 - 4m$	$m^2 - 4m < 15$
-3	$(-3)^2 - 4(-3) = 21$	F
-2	$(-2)^2 - 4(-2) = 12$	T
-1	$(-1)^2 - 4(-1) = 5$	T
0	$(0)^2 - 4(0) = 0$	T
1	$(1)^2 - 4(1) = -3$	T
2	$(2)^2 - 4(2) = -4$	T
3	$(3)^2 - 4(3) = -3$	T
4	$(4)^2 - 4(4) = 0$	T
5	$(5)^2 - 4(5) = 5$	T
6	$(6)^2 - 4(6) = 12$	T
7	$(7)^2 - 4(7) = 21$	F

Therefore, we have

$$A = \{-2, \ -1, \ 0, \ 1, \ 2, \ 3, \ 4, \ 5, \ 6\}$$

Remark: This can also be seen by graphing the function $f(x) = x^2 - 4x$ and the function $g(x) = 15$ on the interval $[-4, \ 7]$:



and checking for which integer values of x the graph of f is below the graph of g. If $x \in \mathbb{Z}$, then $f(x) < 15$ for $x \in \{-2, \ -1, \ 0, \ 1, \ 2, \ 3, \ 4, \ 5, \ 6\}$.

Exercises 5-10: In Exercises $5-10$, decide whether the object described is indeed a set (is well defined). Give an alternative description of each set.

5. $\{n \in \mathbb{Z}^+ \mid n \text{ is a large number}\}$

Solution: The characterizing property $P(n)$: "n is a large number" is not well defined, since the statement "n is a large number" cannot be assigned neither true nor false. Therefore, $\{n \in \mathbb{Z}^+ \mid n \text{ is a large number}\}$ is not a set.

6. $\left\{n \in \mathbb{Z} \mid n^2 < 0\right\}$

Solution: The characterizing property $P(n)$: "$n^2 < 0$" is not true for any integer, therefore

$$\left\{n \in \mathbb{Z} \mid n^2 < 0\right\} = \emptyset, \text{ (}\emptyset \text{ denotes the empty set),}$$

so $\left\{n \in \mathbb{Z} \mid n^2 < 0\right\}$ is a set.

7. $\left\{n \in \mathbb{Z} \mid 39 < n^3 < 57\right\}$

Solution: The characterizing property $P(n)$: "$39 < n^3 < 57$" is not true for any integer, therefore

$$\left\{n \in \mathbb{Z} \mid 39 < n^3 < 57\right\} = \emptyset,$$

so $\left\{n \in \mathbb{Z} \mid 39 < n^3 < 57\right\}$ is a set.

8. $\{x \in \mathbb{Q} \mid x \text{ is almost an integer}\}$

Solution: The characterizing property $P(x)$: "x is almost an integer" is not well defined, since the statement "x is almost an integer" cannot be assigned neither true nor false. Therefore,

$$\{x \in \mathbb{Q} \mid x \text{ is almost an integer}\} \text{ is not a set.}$$

9. $\{x \in \mathbb{Q} \mid x \text{ may be written with denominator greater than } 100\}$

Solution: The characterizing property $P(x)$: "x may be written with denominator greater than 100" is well defined, therefore

$$\{x \in \mathbb{Q} \mid x \text{ may be written with denominator greater than } 100\} \text{ is a set.}$$

We observe that every rational number $x = \frac{m}{n}$, where $m \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$ can be written as a fraction with the denominator greater than 100, that is:

$$\frac{m}{n} = \frac{r}{s}, \text{ for some } r \in \mathbb{Z},\ s \in \mathbb{Z}^+, \text{ and } s > 100.$$

This is because,

- if $n > 100$, we just take $s = n$ and $r = m$, and we are finished.
- if $n \leq 100$, we have:
$$\frac{m}{n} = \frac{101m}{101n}$$
since $n \geq 1$, we have $101n \geq 101$, and we take $s = 101n$ and $r = 101m$.

Therefore, we have:

$$\{x \in \mathbb{Q} \mid x \text{ may be written with denominator greater than } 100\} = \mathbb{Q}.$$

10. $\{x \in \mathbb{Q} \mid x \text{ may be written with positive denominator less than } 4\}$.

Solution: The characterizing property $P(x)$: "x may be written with positive denominator less than 4" is well defined, therefore

$$A = \{x \in \mathbb{Q} \mid x \text{ may be written with positive denominator less than } 4\} \text{ is a set.}$$

To give an alternative description of this set we consider the statement "x may be written with positive denominator less than 4". As we observe, this statement is equivalent with the following statement:

$$\text{"There exists } m \in \mathbb{Z} \text{ and there exists } n \in \{1,2,3\} \text{ such that } x = \frac{m}{n}\text{".}$$

Thus, we can also define the set A as follows:

$$A = \left\{x \in \mathbb{Q} \mid \text{There exists } m \in \mathbb{Z} \text{ and there exists } n \in \{1,2,3\} \text{ such that } x = \frac{m}{n}\right\}$$

Exercise 12: Let $A = \{1,\ 2,\ 3\}$ and $B = \{2,\ 4,\ 6\}$. For each relation between A and B given as a subset of $A \times B$, decide whether it is a function mapping A into B. If it is a function, decide whether it is one to one and whether it is onto B.

a. $\{(1,4),\ (2,4),\ (3,6)\}$

Solution: Let $\mathcal{R} = \{(1,4),\ (2,4),\ (3,6)\}$, since for all $x \in A$, there is unique $y \in B$ such that

$$(x,y) \in \mathcal{R},$$

then $\mathcal{R}$ is a function.

- $\mathcal{R}$ is not one to one function since $(1,4),(2,4) \in \mathcal{R}$ and the second coordinates of these pairs are the same for different values of the first coordinates.
- $\mathcal{R}$ is not onto function since there is no $x \in A$, such that $(x,2) \in \mathcal{R}$.

b. $\{(1,4),\ (2,6),\ (3,4)\}$

Solution: Let $\mathcal{R} = \{(1,4),\ (2,6),\ (3,4)\}$, since for all $x \in A$, there is unique $y \in B$ such that

$$(x,y) \in \mathcal{R},$$

then $\mathcal{R}$ is a function.

- $\mathcal{R}$ is not one to one function since $(1,4),(3,4) \in \mathcal{R}$ and the second coordinates of these pairs are the same for different values of the first coordinates.
- $\mathcal{R}$ is not onto function since there is no $x \in A$, such that $(x,2) \in \mathcal{R}$.

c. $\{(1,6),\ (1,2),\ (1,4)\}$

Solution: Let $\mathcal{R} = \{(1,6),\ (1,2),\ (1,4)\}$. We observe that for $x = 2$ there is no $y \in B$ such that $(2,\ y) \in \mathcal{R}$, therefore the relation $\mathcal{R}$ is not a function.

d. $\{(2,2), \ (1,6), \ (3,4)\}$

Solution: Let $\mathcal{R} = \{(2,2), \ (1,6), \ (3,4)\}$, since for all $x \in A$, there is a unique $y \in B$ such that

$$(x, y) \in \mathcal{R},$$

then $\mathcal{R}$ is a function.

- $\mathcal{R}$ is one to one function since for every $(x_1, y_1), (x_2, y_2) \in \mathcal{R}$ if $y_1 = y_2$ then we have also $x_1 = x_2$.
- $\mathcal{R}$ is onto function since for every $y \in B$ there is $x \in A$, such that $(x, y) \in \mathcal{R}$.

e. $\{(1,6), \ (2,6), \ (3,6)\}$

Solution: Let $\mathcal{R} = \{(1,6), \ (2,6), \ (3,6)\}$, since for all $x \in A$, there is unique $y \in B$ such that

$$(x, y) \in \mathcal{R},$$

then $\mathcal{R}$ is a function.

- $\mathcal{R}$ is not one to one function since $(1,6), (2,6) \in \mathcal{R}$ and the second coordinates of these pairs are the same for different values of the first coordinates. We also observe that $\mathcal{R}$ defines a constant function from A into B.
- $\mathcal{R}$ is not onto function since there is no $x \in A$, such that $(x, 2) \in \mathcal{R}$.

f. $\{(1,2), \ (2,6), \ (2,4)\}$

Solution: Let $\mathcal{R} = \{(1,2), \ (2,6), \ (2,4)\}$. We observe that for $x = 3$ there is no $y \in B$ such that $(3, \ y) \in \mathcal{R}$, therefore the relation $\mathcal{R}$ is not a function.

Exercise 14: Recall that for $a, b \in \mathbb{R}$ and $a < b$, the *closed interval* $[a, \ b]$ in $\mathbb{R}$ is defined by

$$[a, \ b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}.$$

Show that the given intervals have the same cardinality by giving a formula for one-to-one function f mapping the first interval onto the second interval.

a. $[0, \ 1]$ and $[0, \ 2]$

Solution: Let $f : [0, \ 1] \to [0, \ 2]$ be defined as follows:

$$f(x) = 2x, \text{ for all } x \in \mathbb{R}.$$

We observe that f is one-to-one, since for every $x_1, x_2 \in [0, \ 1]$: if $f(x_1) = f(x_2)$ then

$$\begin{aligned} 2x_1 &= 2x_2, \text{ so} \\ x_1 &= x_2. \end{aligned}$$

The function f is also onto $[0, 2]$, since for any $y \in [0, \ 2]$, there is $x = \frac{y}{2} \in [0, 1]$ such that:

$$f(x) = f\left(\frac{y}{2}\right) = 2\left(\frac{y}{2}\right) = y.$$

b. $[1,\ 3]$ and $[5,\ 25]$

Solution: We need to find a function $f:[1,\ 3]\to[5,\ 25]$ such that f is one-to-one and onto. However, there are infinitely many such functions, the simplest such a function that could be defined is the polynomial function of degree 1, that is the function defined by:

$$f(x)=ax+b,\ \text{ where } a,b\in\mathbb{R}.$$

In order to find such function we assume that

$$f(1)=5 \text{ and } f(3)=25.$$

Therefore, we have:

$$f(1)=a+b=5 \text{ and } f(3)=3a+b=25,$$

hence we need to solve the following system of linear equations for a and b :

$$\begin{cases} a & + & b & = & 5 \\ 3a & + & b & = & 25 \end{cases}$$

We have

$$\begin{aligned}(a+b)-(3a+b) &= 5-25\\ -2a &= -20\\ a &= 10\end{aligned}$$

and since

$$\begin{aligned}a+b &= 5 \text{ and } a=10,\ \text{so}\\ 10+b &= 5\\ b &= -5\end{aligned}$$

Therefore, our function is given by:

$$f(x)=10x-5$$

We observe that f is one-to-one, since for every $x_1,x_2\in[1,\ 3]$: if $f(x_1)=f(x_2)$ then

$$\begin{aligned}10x_1-5 &= 10x_2-5,\ \text{so}\\ 10x_1 &= 10x_2\\ x_1 &= x_2.\end{aligned}$$

The function f is also onto $[5,25]$, since for any $y\in[5,\ 25]$, we have

$$\begin{aligned}10x-5 &= y\\ 10x &= y+5\\ x &= \frac{y+5}{10}\end{aligned}$$

Since $y\in[5,\ 25]$, we have:

$$1=\frac{5+5}{10}\le\frac{y+5}{10}\le\frac{25+5}{10}=3$$

it follows that $x=\frac{y+5}{10}\in[1,3]$ and

$$f(x)=f\left(\frac{y+5}{10}\right)=10\left(\frac{y+5}{10}\right)-5=y \text{ as required.}$$

c. $[a, b]$ and $[c, d]$

Solution: We need to find a function $f : [a,\ b] \to [c,\ d]$ such that f is one-to-one and onto. However, there are infinitely many such functions, the simplest such a function that could be defined is the polynomial function of degree 1, that is the function defined by:

$$f(x) = mx + n, \text{ where } n, m \in \mathbb{R}.$$

In order to find such function we assume that

$$f(a) = c \text{ and } f(b) = d.$$

Therefore, we have:

$$f(a) = ma + n = c \text{ and } f(b) = mb + n = d,$$

hence we need to solve the following system of linear equations for n and m :

$$\begin{cases} am & + & n & = & c \\ bm & + & n & = & d \end{cases}$$

We have

$$\begin{aligned} (am + n) - (bm + n) &= c - d \\ (a - b)\,m &= c - d \\ m &= \frac{c - d}{a - b} \end{aligned}$$

and since

$$\begin{aligned} am + n &= c \text{ and } m = \frac{c - d}{a - b}, \text{ so} \\ a\frac{c - d}{a - b} + n &= c \\ n &= c - \frac{a(c - d)}{a - b} \end{aligned}$$

Therefore, our function is given by:

$$f(x) = \left(\frac{c - d}{a - b}\right) x + c - \frac{a(c - d)}{a - b}$$

We observe that f is one-to-one, since for every $x_1, x_2 \in [a,\ b]$: if $f(x_1) = f(x_2)$ then

$$\begin{aligned} \left(\frac{c - d}{a - b}\right) x_1 + c - \frac{a(c - d)}{a - b} &= \left(\frac{c - d}{a - b}\right) x_2 + c - \frac{a(c - d)}{a - b}, \text{ so} \\ \left(\frac{c - d}{a - b}\right) x_1 &= \left(\frac{c - d}{a - b}\right) x_2 \\ x_1 &= x_2. \end{aligned}$$

The function f is also onto $[c, d]$, since for any $y \in [c,\ d]$, we have

$$\begin{aligned} \left(\frac{c - d}{a - b}\right) x + c - \frac{a(c - d)}{a - b} &= y \\ \left(\frac{c - d}{a - b}\right) x &= y + \frac{a(c - d)}{a - b} - c \\ x &= \frac{y + \frac{a(c-d)}{a-b} - c}{\frac{c-d}{a-b}} = -\frac{ad - bc - ay + by}{c - d} \end{aligned}$$

Since $y \in [c,\ d]$, we have:

$$a = -\frac{ad - bc - ac + bc}{c - d} \leq -\frac{ad - bc - ay + by}{c - d} \leq -\frac{ad - bc - ad + bd}{c - d} = b$$

it follows that $x = -\frac{ad-bc-ay+by}{c-d} \in [a, b]$ and

$$f(x) = f\left(-\frac{ad - bc - ay + by}{c - d}\right) = \left(\frac{c - d}{a - b}\right)\left(-\frac{ad - bc - ay + by}{c - d}\right) + c - \frac{a(c - d)}{a - b} = y \text{ as required.}$$

Exercise 16-19: For any set A we denote by $\mathcal{P}(A)$ the collection of all subsets of A. For example, if $A = \{a, b, c, d\}$, then $\{a, b, c\} \in \mathcal{P}(A)$. Then set $\mathcal{P}(A)$ is the *power set* A.

16. List the elements of the power set of the given set and give the cardinality of the power set.

a. $\emptyset$

Solution: $\mathcal{P}(\emptyset) = \{\emptyset\}$ and $|\mathcal{P}(\emptyset)| = |\{\emptyset\}| = 1$.

b. $\{a\}$

Solution: $\mathcal{P}(\{a\}) = \{\emptyset,\ \{a\}\}$ and $|\mathcal{P}(\{a\})| = |\{\emptyset,\ \{a\}\}| = 2$.

c. $\{a, b\}$

Solution: $\mathcal{P}(\{a, b\}) = \{\emptyset,\ \{a\}, \{b\}, \{a, b\}\}$ and $|\mathcal{P}(\{a, b\})| = |\{\emptyset,\ \{a\}, \{b\}, \{a, b\}\}| = 4$.

d. $\{a, b, c\}$

Solution: $\mathcal{P}(\{a, b, c\}) = \{\emptyset,\ \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$ and

$$|\mathcal{P}(\{a, b, c\})| = |\{\emptyset,\ \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}| = 8.$$

17. Let A be a finite set, and let $|A| = s$. Based on the preceding exercise, make a conjecture about the value of $|\mathcal{P}(A)|$. Then try to prove your conjecture.

Solution: We observe that:

- $|\mathcal{P}(\emptyset)| = 2^0 = 2^{|\emptyset|}$,
- $|\mathcal{P}(\{a\})| = 2^1 = 2^{|\{a\}|}$,
- $|\mathcal{P}(\{a, b\})| = 2^2 = 2^{|\{a,b\}|}$,
- $|\mathcal{P}(\{a, b, c\})| = 2^3 = 2^{|\{a,b,c\}|}$

Therefore, we make the following conjecture:

Conjecture 0.1 *If* $|A| = s$ *then* $|\mathcal{P}(A)| = 2^s$.

Proof. We observe that given a subset $B \subseteq A$, we can define a function

$$\begin{aligned} \chi_{A,B} &: A \to \{0, 1\} \text{ as follows:} \\ \chi_{A,B}(x) &= \begin{cases} 1 & \text{if} \quad x \in B \\ 0 & \text{if} \quad x \notin B \end{cases} \end{aligned}$$

If $A = \{a_1, a_2, ..., a_s\}$, then for each subset $B \subseteq A$, we assign the following binary sequence:

$$B \longleftrightarrow \left(\chi_{A,B}(a_1), \chi_{A,B}(a_2), ..., \chi_{A,B}(a_s)\right)$$

The relation defined just above is one-to-one and onto, since given a binary sequence $r = (r_1, r_2, ..., r_s)$ we define the subset $B \subseteq A$ via the following:

$$B = \{a_i \in A \mid r_i = 1,\ i = 1, 2, ..., s\}$$

Therefore, if $X = \{(r_1, r_2, ..., r_s) \mid r_i \in \{0, 1\},\ i = 1, 2, ..., s\}$ is the set of all binary sequences of length s, we have

$$|X| = |\mathcal{P}(A)|$$

Obviously, we have

$$|X| = \underbrace{2 \cdot 2 \cdot ... \cdot 2}_{s \text{ factors}} = 2^s$$

since on each position of the sequence $(r_1, r_2, ..., r_s)$ we either put 0 or 1. It follows that

$$|\mathcal{P}(A)| = 2^s$$

as we claimed. **Therefore, our Conjecture is a Theorem!** ■

18. For any set A, finite or infinite, let B^A be the set of all functions mapping A into the set $B = \{0, 1\}$. Show that the cardinality of B^A is the same as the cardinality of the set $\mathcal{P}(A)$.

Solution: We observe that given a subset $C \subseteq A$, we can define a function

$$\begin{aligned} \chi_{A,C} &: A \to \{0,1\} \text{ as follows:} \\ \chi_{A,C}(x) &= \begin{cases} 1 & \text{if} \quad x \in C \\ 0 & \text{if} \quad x \notin C \end{cases} \end{aligned}$$

Then, there is a natural one-to-one and onto correspondence between elements of $\mathcal{P}(A)$ and elements of B^A, that we define as follows:

$$\begin{aligned} \phi &: \mathcal{P}(A) \to B^A, \\ \phi(C) &= \chi_{A,C} \end{aligned}$$

Obviously, ϕ is injective, since for any $C, D \in \mathcal{P}(A)$ if $\phi(C) = \phi(D)$, then

$$\begin{aligned} \chi_{A,C} &= \chi_{A,D}, \text{ thus} \\ \chi_{A,C}(x) &= \chi_{A,D}(x) \text{ for all } x \in A \end{aligned}$$

the last statement, by the definition of $\chi_{A,C}$ and $\chi_{A,D}$, implies that $C = D$.

Now, we show that ϕ is onto. Let $\chi \in B^A$, we define $C \in \mathcal{P}(A)$ as follows:

$$C = \{x \in A \mid \chi(x) = 1\}$$

as we see

$$\chi = \chi_{A,C}$$

and we have

$$\phi(C) = \chi_{A,C} = \chi.$$

It follows that ϕ is bijection, so $|\mathcal{P}(A)| = \left|B^A\right|$.

19. Show that the power set of a set A, finite or infinite, has too many elements to be able to be put in a one-to-one correspondence with A. Explain why this intuitively means that there are an infinite number of infinite cardinal numbers.

Solution: Suppose that

$$\phi : A \to \mathcal{P}(A)$$

is an one-to-one function. We show that ϕ cannot be onto. We define the following subset $S \subseteq A$:

$$S = \{x \in A \mid x \notin \phi(x)\}.$$

We show that for every $y \in A$, we have

$$\phi(y) \neq S.$$

Suppose that there is $y \in A$ such that

$$\phi(y) = S$$

- If $y \in S$ then
$$y \in \{x \in A \mid x \notin \phi(x)\} \text{ if and only if } y \in A \text{ and } y \notin \phi(y) = S,$$
so we have $y \in S$ if and only if $y \notin S$, contradiction.
- If $y \notin S$, then since $y \in A$ and $y \notin S = \phi(y)$, so $y \in A$ and $y \notin \phi(y)$, thus by the definition of S it follows that $y \in S$, again contradiction.

Therefore, we have that

$$\phi(y) \neq S, \text{ for all } y \in A.$$

This shows that there is no bijection between A and $\mathcal{P}(A)$. We then conclude that

$$|A| < |\mathcal{P}(A)|.$$

Now, we let $A = \mathbb{Z}$, then we have:

$$\aleph_0 = |\mathbb{Z}| < |\mathcal{P}(\mathbb{Z})| < |\mathcal{P}(\mathcal{P}(\mathbb{Z}))| < ...$$

It follows that there are an infinite number of infinite cardinal numbers.

Exercise 23-27: In Exercises 23 through 27, find the number of different partitions of a set having the given number of elements.

23. 1 element set.

Solution: Let $A = \{a\}$, then $|A| = 1$. Since any partition of A must consists of nonempty subsets of A, thus we have just one partition of A, that is $\{\{a\}\}$.

24. 2 element set.

Solution: Let $A = \{a, b\}$, then $|A| = 2$. We have only two partitions of A :

$$\begin{aligned} P_1 &= \{\{a\}, \{b\}\} \\ P_2 &= \{\{a, b\}\}. \end{aligned}$$

25. 3 element set.

Solution: Let $A = \{a, b, c\}$, then $|A| = 3$. We have 5 partitions of A :

$$
\begin{aligned}
P_1 &= \{\{a\}, \{b\}, \{c\}\} \\
P_2 &= \{\{a\}, \ \{b, c\}\} \\
P_3 &= \{\{b\}, \ \{a, c\}\} \\
P_4 &= \{\{c\}, \ \{a, b\}\} \\
P_5 &= \{\{a, b, c\}\}
\end{aligned}
$$

26. 4 element set.

Solution: Let $A = \{a, b, c, d\}$, then $|A| = 4$. We have 15 partitions of A :

$$
\begin{array}{lll}
P_1 = \{\{a\}, \{b\}, \{c\}, \{d\}\} & P_2 = \{\{a\}, \ \{b, c\}, \{d\}\} & P_3 = \{\{b\}, \ \{a, c\}, \{d\}\} \\
P_4 = \{\{c\}, \ \{a, b\}, \{d\}\} & P_5 = \{\{a, b, c\}, \{d\}\} & P_6 = \{\{a, d\}, \{b\}, \{c\}\} \\
P_7 = \{\{a\}, \{b, d\}, \{c\}\} & P_8 = \{\{a\}, \{b\}, \{c, d\}\} & P_9 = \{\{a, d\}, \ \{b, c\}\} \\
P_{10} = \{\{a\}, \ \{b, c, d\}\} & P_{11} = \{\{b, d\}, \ \{a, c\}\} & P_{12} = \{\{b\}, \ \{a, c, d\}\} \\
P_{13} = \{\{c, d\}, \ \{a, b\}\} & P_{14} = \{\{c\}, \ \{a, b, d\}\} & P_4 = \{\{a, b, c, d\}\}
\end{array}
$$

27. 5 element set.

Solution: Let $A = \{a, b, c, d, e\}$, then $|A| = 5$. To find the total number of all partitions of A, we use the partitions of the 4 element set in two ways.

- We insert $\{e\}$ as a separate cell into the partition of 4 element set, for instance:

  $$P = \{\{a, b, c\}, \{d\}, \{e\}\} \text{ is obtained from } P_3 \text{ by inserting } \{e\} \text{ as a separate cell.}$$

  We obtain 15 partitions of A is such a way.

- We insert the element $e \in A$ into one of the cells of the existing partition of the 4 element set, for instance

  $$P = \{\{a, e\}, \{b\}, \{c, d\}\} \text{ is obtained by inserting } e \text{ into the cell } \{a\} \text{ in the partition } P_8.$$

  Therefore, we have:

  Since there is only one partition of the set of 4 elements with 4 cells, we obtain $4 \cdot 1 = 4$ new partitions of A.

  Since there are 6 partitions of the set of 4 elements with 3 cells, we obtain $6 \cdot 3 = 18$ new partitions of A.

  Since there are 7 partitions of the set of 4 elements with 2 cells, we obtain $7 \cdot 2 = 14$ new partitions of A.

  Since there is only one partition of the set of 4 elements with 1 cell, we obtain $1 \cdot 1 = 1$ new partitions of A.

Therefore, we have the following total number of partitions of the 5 element set:

$$15 + 4 + 18 + 14 + 1 = 52.$$

Exercise 29-34: In Exercises 29 through 34, determine whether the given relation is an equivalence relation on the set. Describe the partition arising from each equivalence relation. Recall, that $\mathcal{R} \subseteq X \times X$ is called an equivalence relation if for all $x, y, z \in X$, $\mathcal{R}$ satisfies the following properties:

a. $x\mathcal{R}x$

b. $x\mathcal{R}y$ if and only if $y\mathcal{R}x$

c. If $x\mathcal{R}y$ and $y\mathcal{R}z$, then $x\mathcal{R}z$.

29. $n\mathcal{R}m$ in $\mathbb{Z}$ if $nm > 0$.

Solution: We verify the conditions $(a) - (c)$ in the definition given above:

a. Let $n \in \mathbb{Z}\setminus\{0\}$, then $nn = n^2 > 0$, so $n\mathcal{R}n$ for $n \in \mathbb{Z}\setminus\{0\}$. However, for $n = 0$, it is not true that $0\mathcal{R}0$, since $0 \cdot 0 > 0$ is not true. Therefore, $\mathcal{R}$ is not reflexive, and it follows that $\mathcal{R}$ is not an equivalence relation.

30. $x\mathcal{R}y$ in $\mathbb{R}$ if $x \geq y$.

Solution: We verify the conditions $(a) - (c)$ in the definition given above:

a. Let $x \in \mathbb{R}$, then $x \geq x$, so $x\mathcal{R}x$ for every $x \in \mathbb{R}$. Therefore, $\mathcal{R}$ is reflexive.

b. Let $x = 4$ and $y = 2$, then $x\mathcal{R}y$ since $4 \geq 2$, however $4 \nleq 2$, so $y\mathcal{R}x$ is not true. Therefore, $\mathcal{R}$ is not reflexive.

Since $\mathcal{R}$ is not reflexive, then $\mathcal{R}$ is not an equivalence relation.

31. $x\mathcal{R}y$ in $\mathbb{R}$ if $|x| = |y|$.

Solution: We verify the conditions $(a) - (c)$ in the definition given above:

a. Let $x \in \mathbb{R}$, then $|x| = |x|$, so $x\mathcal{R}x$ for all $x \in \mathbb{R}$. Therefore, $\mathcal{R}$ is reflexive.

b. Let $x, y \in \mathbb{R}$ and $x\mathcal{R}y$. Therefore, $|x| = |y|$, since " $=$ " relation is symmetric thus $|y| = |x|$, and it follows that $y\mathcal{R}x$. This shows that $x\mathcal{R}y$ if and only if $y\mathcal{R}x$, so $\mathcal{R}$ is symmetric.

c. Let $x, y, z \in \mathbb{R}$ and assume that $x\mathcal{R}y$ and $y\mathcal{R}z$. It follows that

$$|x| = |y| \text{ and } |y| = |z|,$$

since " $=$ " relation is transitive, then

$$|x| = |z|, \text{ and it follows by the definition of } \mathcal{R} \text{ that } x\mathcal{R}z.$$

Thus, $\mathcal{R}$ is transitive.

Since $\mathcal{R}$ satisfies all conditions $(a) - (c)$, we conclude that $\mathcal{R}$ is an equivalence relation.
Now, for any $x \in \mathbb{R}$, we have:

$$\begin{aligned} \overline{x} &= \{y \in \mathbb{R} \mid x\mathcal{R}y\} = \{y \in \mathbb{R} \mid |x| = |y|\} \\ &= \{y \in \mathbb{R} \mid y = \pm |x|\} = \{y \in \mathbb{R} \mid y = \pm x\} \\ &= \{-x,\ x\}. \end{aligned}$$

32. $x\mathcal{R}y$ in $\mathbb{R}$ if $|x-y| \leq 3$.

Solution: We verify the conditions $(a)-(c)$ in the definition given above:

a. Let $x \in \mathbb{R}$, then $|x-x| = 0 \leq 3$, so $x\mathcal{R}x$ is true for all $x \in \mathbb{R}$. Hence, $\mathcal{R}$ is reflexive.

b. Let $x, y \in \mathbb{R}$ and assume that $x\mathcal{R}y$, that is $|x-y| \leq 3$. Since $|x-y| = |-(y-x)| = |y-x| \leq 3$, then also $y\mathcal{R}x$ is true, so $\mathcal{R}$ is symmetric.

c. Let $x = 1$, $y = 4$, $z = 7$, then $|x-y| = 3 \leq 3$ and $|y-z| = 3 \leq 3$, so $x\mathcal{R}y$ and $y\mathcal{R}z$ are true, but

$$|x-z| = |1-7| = 6 \nleq 3, \text{ so}$$

$x\mathcal{R}z$ is not true. Therefore, $\mathcal{R}$ is not transitive.

Since $\mathcal{R}$ is not transitive, $\mathcal{R}$ is not an equivalence relation.

33. $n\mathcal{R}m$ in $\mathbb{Z}^+$ if n and m have the same number of digits in the usual base ten notation.

Solution: We verify the conditions $(a)-(c)$ in the definition given above. Let $n = a_1a_2...a_k$, where $a_i \in \{0, 1, 2, ..., 9\}$, $i = 2, 3, .., k$, and $a_1 \in \{1, 2, ..., 9\}$, where $k \in \mathbb{Z}^+$. Let

$$\begin{aligned} N(n) &= \textit{number of digits in the ussual base ten notation}, \text{ that is,} \\ \textit{if } n &= a_1a_2...a_k, \text{ then } N(n) = N(a_1a_2...a_k) = k. \end{aligned}$$

a. Then $n\mathcal{R}n$ since $N(n) = N(n)$.

b. Let $n = a_1a_2...a_k$ and $m = b_1b_2...b_l$ be positive integers. If $x\mathcal{R}y$ then $N(n) = N(m)$, therefore $k = l$. Since the relation "$=$" is symmetric, then $N(n) = N(m)$ implies that $N(m) = N(n)$, thus by the definition of $\mathcal{R}$, we have $m\mathcal{R}n$.

c. Let $n, m, p \in \mathbb{Z}^+$, and assume that $n\mathcal{R}m$ and $m\mathcal{R}p$. Therefore, we have

$$N(n) = N(m) \text{ and } N(m) = N(p).$$

Thus, we have by transitivity of "$=$" :

$$N(n) = N(p).$$

Thus, by the definition of $\mathcal{R}$, we have that $n\mathcal{R}p$.

Since the conditions $(a)-(c)$ are satisfied, then $\mathcal{R}$ is an equivalence relation.
Now, we find the equivalence classes of $\mathcal{R}$ in $\mathbb{Z}^+$. Let $n \in \mathbb{Z}^+$, then $N(n) = k$, for $k \in \mathbb{Z}^+$, and we have:

$$\begin{aligned} \overline{n} &= \left\{m \in \mathbb{Z}^+ \mid n\mathcal{R}m\right\} = \left\{m \in \mathbb{Z}^+ \mid N(n) = N(m)\right\} \\ &= \left\{m \in \mathbb{Z}^+ \mid N(m) = k\right\} = \left\{ 1\underbrace{00...0}_{k-1 \text{ zeros}},\ 100...1, ..., \underbrace{999...9}_{k \text{ digits}} \right\} \end{aligned}$$

Therefore, $\mathcal{R}$ partitions $\mathbb{Z}^+$ into the following cells:

$$\begin{aligned} \overline{1} &= \{1, 2, ..., 9\} \text{ one digit numbers} \\ \overline{10} &= \{10, 11, ..., 99\} \text{ two digit numbers} \\ \overline{100} &= \{100, 101, ..., 999\} \text{ three digit numbers.} \\ &\vdots \\ \overline{1\underbrace{00...0}_{k-1 \text{ zeros}}} &= \left\{ 1\underbrace{00...0}_{k-1 \text{ zeros}},\ 100...1, ..., \underbrace{999...9}_{k \text{ digits}} \right\} k \text{ digit numbers} \\ &\vdots \end{aligned}$$

34. $n\mathcal{R}m$ in $\mathbb{Z}^+$ if n and m have the same final digit in the usual base ten notation.

Solution: We verify the conditions $(a)-(c)$ in the definition given above. Let $n = a_1a_2...a_k$, where $a_i \in \{0,1,2,...,9\}$, $i = 2,3,..,k$, and $a_1 \in \{1,2,...,9\}$, where $k \in \mathbb{Z}^+$. Let

$$\begin{aligned} \widehat{N}(n) &= \textit{final digit of } n \textit{ in the ussual base ten notation}, \text{ that is,} \\ \textit{if } n &= a_1a_2...a_k, \text{ then } \widehat{N}(n) = \widehat{N}(a_1a_2...a_k) = a_k. \end{aligned}$$

a. Then $n\mathcal{R}n$ since $\widehat{N}(n) = \widehat{N}(n)$.

b. Let $n = a_1a_2...a_k$ and $m = b_1b_2...b_l$ be positive integers. If $x\mathcal{R}y$ then $\widehat{N}(n) = \widehat{N}(m)$, therefore $a_k = b_l$. Since the relation " = " is symmetric, then $\widehat{N}(n) = \widehat{N}(m)$ implies that $\widehat{N}(m) = \widehat{N}(n)$, thus by the definition of $\mathcal{R}$ we have $m\mathcal{R}n$.

c. Let $n, m, p \in \mathbb{Z}^+$, and assume that $n\mathcal{R}m$ and $m\mathcal{R}p$. Therefore, we have

$$\widehat{N}(n) = \widehat{N}(m) \text{ and } \widehat{N}(m) = \widehat{N}(p).$$

Thus, we have by transitivity of " = " :

$$\widehat{N}(n) = \widehat{N}(p).$$

Thus, by the definition of $\mathcal{R}$, we have that $n\mathcal{R}p$.

Since the conditions $(a)-(c)$ are satisfied, then $\mathcal{R}$ is an equivalence relation.
Now, we find the equivalence classes of $\mathcal{R}$ in $\mathbb{Z}^+$. Let $n = a_1a_2...a_k \in \mathbb{Z}^+$, then $\widehat{N}(n) = a_k$, $a_k \in \{0,1,2,...,9\}$ for $k \geq 2$, and $a_k \in \{1,2,...,9\}$ if $k = 1$. We have:

$$\begin{aligned} \overline{n} &= \{m \in \mathbb{Z}^+ \mid n\mathcal{R}m\} = \left\{m \in \mathbb{Z}^+ \mid \widehat{N}(n) = \widehat{N}(m)\right\} \\ &= \left\{m \in \mathbb{Z}^+ \mid \widehat{N}(m) = a_k, \text{ and } a_k \in \{0,1,2,...,9\}\right\} \end{aligned}$$

Therefore, $\mathcal{R}$ partitions $\mathbb{Z}^+$ into the following cells:

$$\begin{aligned} \overline{1} &= \{1, 11, 21, ..., 91, 101, 121, ..991, ...\} \\ \overline{2} &= \{2, 12, 22, ..., 92, 102, 122, ..992, ...\} \\ \overline{3} &= \{3, 13, 23, ..., 93, 103, 123, ..993, ...\} \\ &\vdots \\ \overline{9} &= \{9, 19, 29, ..., 99, 109, 129, ..999, ...\} \\ \overline{10} &= \{10, 20, 30, ...90, 100, 110, .., 990, ...\} \end{aligned}$$

36. Let $n \in \mathbb{Z}^+$ and let $\sim$ be defined on $\mathbb{Z}$ by $r \sim s$ if and only if $r - s$ is divisible by n, that is, if and only if,

$$r - s = nq \text{ for some } q \in \mathbb{Z}.$$

a. Show that $\sim$ is an equivalence relation defined on $\mathbb{Z}$.

Solution: We verify the conditions $(a)-(c)$ in the definition given above.

a. Let $r \in \mathbb{Z}$, then $r - r = 0$ and 0 is divisible by n. Therefore, $r \sim r$, and $\sim$ is reflexive.

b. Let $r, s \in \mathbb{Z}$, and assume that $r \sim s$, thus there is $q \in \mathbb{Z}$, such that

$$r - s = nq \text{ for some } q \in \mathbb{Z}.$$

We have

$$s - r = -(r - s) = -qn = (-q)\, n.$$

Let $k = -q$, since $q \in \mathbb{Z}$, then $k \in \mathbb{Z}$ and we have

$$s - r = kn, \text{ for some } k \in \mathbb{Z}.$$

Hence, $s - r$ is divisible by n. By the definition of $\sim$, it follows that $s \sim r$, so $\sim$ is symmetric.

c. Let $r, s, t \in \mathbb{Z}$, and assume that $r \sim s$ and $s \sim t$. By the definition of $\sim$, we have:

$$\begin{aligned} r - s &= nq \text{ for some } q \in \mathbb{Z}, \text{ and} \\ s - t &= np \text{ for some } p \in \mathbb{Z}. \end{aligned}$$

Thus, we have

$$\begin{aligned} (r - s) + (s - t) &= nq + np \\ r - t &= n(q + p), \end{aligned}$$

Let $k = q + p$, since $p, q \in \mathbb{Z}$, then $q + p = k \in \mathbb{Z}$, and we have:

$$r - t = nk, \text{ for some } k \in \mathbb{Z}.$$

Now, by the definition $\sim$, we have that $r \sim t$. Thus, the relation $\sim$ is transitive.

Since $\sim$ satisfies conditions $(a) - (c)$, we have that $\sim$ is an equivalence relation.

b. Show that, when restricted to the set $\mathbb{Z}^+ \subset \mathbb{Z}$, this $\sim$ is the equivalence relation, *congruence modulo* n.

Solution: Let $r, s \in \mathbb{Z}^+$. Then $r \sim s$ if and only if $r - s = nq$ for some $q \in \mathbb{Z}$. Now, recall that

$$r \equiv s \pmod{n}$$

if and only if

$$\begin{aligned} r &= nq_1 + r_1 \\ s &= nq_2 + r_2, \text{ and } r_1 = r_2, \text{ where} \\ 0 &\le r_1 \le n - 1, \text{ and } 0 \le r_2 \le n - 1 \end{aligned}$$

Therefore, we have: if $r \equiv s \pmod{n}$ then

$$r - s = (nq_1 + r_1) - (nq_2 + r_2) = n(q_1 - q_2) + (r_1 - r_2).$$

Since $r_1 = r_2$, we have

$$r - s = n(q_1 - q_2).$$

Hence, $r - s$ is divisible by n. It follows from the definition of the relation " $\sim$ " that $r \sim s$.
Now, let

$$\begin{aligned} r &= nq_1 + r_1 \\ s &= nq_2 + r_2, \text{ where} \\ 0 &\le r_1 \le n - 1, \text{ and } 0 \le r_2 \le n - 1 \end{aligned}$$

if $r \sim s$ then $r - s = nq$ for some $q \in \mathbb{Z}$, and we have:

$$r - s = (nq_1 + r_1) - (nq_2 + r_2) = n(q_1 - q_2) + (r_1 - r_2) = nq + 0,$$

thus, it follows that

$$r_1 = r_2.$$

Since $r_1 = r_2$, it follows that $r \equiv s \pmod{n}$ as desired.
Therefore, we showed that

$$\text{For all } r, s \in \mathbb{Z}^+,\ r \equiv s \pmod{n} \text{ if and only if } r \sim s.$$

c. The cells of this partition of $\mathbb{Z}$ are *residue classes modulo* n in $\mathbb{Z}$. Find all the residue classes of $\sim$ in $\mathbb{Z}$.

Solution: Let $k \in \mathbb{Z}$. We have, there is $p \in \mathbb{Z}$ such that:

$$k = np + r,\ \text{where } 0 \leq r \leq n - 1$$

by the definition of the cell, we have:

$$\begin{aligned}
\overline{k} &= \{m \in \mathbb{Z} \mid m \sim k\} = \{m \in \mathbb{Z} \mid m - k = nq,\ \text{for some } q \in \mathbb{Z}\} = \{m \in \mathbb{Z} \mid m = k + nq,\ \text{for some } q \in \mathbb{Z}\} \\
&= \{m \in \mathbb{Z} \mid m = np + nq + r,\ \text{for some } q, p \in \mathbb{Z}\} = \{m \in \mathbb{Z} \mid m = n(p+q) + r,\ \text{for some } q, p \in \mathbb{Z}\} \\
&= \{nl + r \mid l \in \mathbb{Z}\} = \{r + nl \mid l \in \mathbb{Z}\}.
\end{aligned}$$

Therefore, we have:

$$\begin{aligned}
\overline{0} &= \{nm \mid m \in \mathbb{Z}\},\ \text{since } 0 = 0n + 0 \\
\overline{-1} &= \{(n-1) + mn \mid m \in \mathbb{Z}\},\ \text{since } -1 = (-1)n + (n-1) \\
\overline{1} &= \{1 + mn \mid m \in \mathbb{Z}\},\ \text{since } 1 = 0n + 1 \\
&\vdots \\
\overline{-(n-1)} &= \{1 + mn \mid m \in \mathbb{Z}\},\ \text{since } -(n-1) = (-1)n + 1 \\
\overline{(n-1)} &= \{(n-1) + mn \mid m \in \mathbb{Z}\},\ \text{since } (n-1) = 0 \cdot n + (n-1) \\
\overline{(-n)} &= \{nm \mid m \in \mathbb{Z}\},\ \text{since } -n = (-1)n + 0 \\
\overline{n} &= \{nm \mid m \in \mathbb{Z}\},\ \text{since } n = 1 \cdot n + 0
\end{aligned}$$

In particular, we have:

$$\overline{0} = \overline{n} = \overline{(-n)},\ \overline{-1} = \overline{(n-1)},\ \overline{1} = \overline{-(n-1)},\ ...,\ \overline{(-k)} = \overline{(n-k)},\ \overline{k} = \overline{-(n-k)},\ ...$$

Therefore, we have the following equivalence classes:

$$\overline{k} = \{k + nm \mid m \in \mathbb{Z}\},\ k = 0, 1, 2, ..., n-1.$$